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Optimal constants in the Marcinkiewicz–Zygmund inequalities

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ABSTRACT

We give the optimal constants in the Marcinkiewicz–Zygmund inequalities for symmetric summands. As an application we substantially improve the estimates of Ren and Liang (2001) in the Marcinkiewicz–Zygmund–Hölder inequality and identify the best possible constants in the symmetric case.

1. Introduction and main results

Let $X_1, \ldots, X_n$ be $n \in \mathbb{N}$ independent and centered real random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{E}[|X_i|^p] < \infty$ for every $i \in \{1, \ldots, n\}$ and for some $p > 0$. Then the Marcinkiewicz–Zygmund inequality says that for every $p \geq 1$ there exist positive constants A_p and B_p depending only on p such that

$$A_p\, \mathbb{E}\left[\left(\sum_{i=1}^n X_i^2\right)^{p/2}\right] \leq \mathbb{E}\left[\left|\sum_{i=1}^n X_i\right|^p\right] \leq B_p\, \mathbb{E}\left[\left(\sum_{i=1}^n X_i^2\right)^{p/2}\right]. \tag{1.1}$$

If the X_i are actually *symmetric*, then the validity of (1.1) can be extended to all $p > 0$, see Chow and Teicher (1997, Theorem 2, p. 386). In this short note we give the *best possible constants* A_p and B_p, for which the Marcinkiewicz–Zygmund inequality is true in case of symmetric summands. More precisely, consider for every fixed $p > 0$ the set $\mathbf{A}_p$ of all $A > 0$ such that

$$A\, \mathbb{E}\left[\left(\sum_{i=1}^n X_i^2\right)^{p/2}\right] \leq \mathbb{E}\left[\left|\sum_{i=1}^n X_i\right|^p\right]$$

for all $n \in \mathbb{N}$ and for all independent, symmetric and p-fold integrable $X_1, \ldots, X_n$. Note that by (1.1) the set $\mathbf{A}_p$ is non-empty and a closed left half-line, whence the *optimal* lower constant $A_{p,opt} := \max \mathbf{A}_p$ exists. Analogously, the optimal upper constant $B_{p,opt}$ exists and is the minimal $B > 0$ such that

$$\mathbb{E}\left[\left|\sum_{i=1}^n X_i\right|^p\right] \leq B\, \mathbb{E}\left[\left(\sum_{i=1}^n X_i^2\right)^{p/2}\right]$$

for all $n \in \mathbb{N}$ and for all independent, symmetric and p-fold integrable $X_1, \ldots, X_n$.

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The proof of (1.1) combines a standard technique known as *symmetrization* with the Khintchine inequalities, see Chow and Teicher (1997, p. 386). Even though the Marcinkiewicz–Zygmund inequalities are well-known and have been extended in many directions, see de la Peña and Giné (1999, Chapter 1), the optimal constants are still unknown. In applications usually only the upper inequality is used, for example in the proof of Strong Laws of Large Numbers for arrays of random variables, see Chow and Teicher (1997, Example 1, p. 393). As for the upper constant the derivation of Chow and Teicher (1997, p. 385) yields

$$B_{p,opt} \leq \bar{p}^{\,p/2} \leq (p+2)^{p/2} \quad \forall p \geq 2,$$

where $\bar{p}$ is the smallest even integer greater or equal to p. Using Theorem 1.3.1 of de la Peña and Giné (1999) this can be easily improved to

$$B_{p,opt} \leq (p-1)^{p/2} \quad \forall p \geq 2. \tag{1.2}$$

Here, de la Peña and Giné (1999, p. 16) point out that these estimates are of the right order as p tends to infinity, but they are not the best possible. Indeed, Egorov (1997, Theorem 1, p. 2) and Egorov (2009, Theorem 1, p. 305) finds that

$$B_{p,opt} \leq Ce^{-p/2}p^{p/2} \quad \forall p \geq 2, \tag{1.3}$$

where C is an absolute constant with $C \geq \sqrt{2}$. That means that his estimate is smaller than the estimate in (1.2) by a factor $Ce^{-(p-1)/2}(1+o(1))$ as $p \to \infty$. However, his proof requires some clarification. To see this note that in a first step Egorov (1997, p. 3) derives his upper bound for all even integers $p \in 2\mathbb{N}$ and in a second step simply refers to the monotonicity of the L_p-norm. But according to Chow and Teicher (1997, p. 385), this monotonicity argument only ensures the weaker relation

$$B_{p,opt} \leq C^{p/\bar{p}}\, e^{-p/2}\bar{p}^{\,p/2} \leq Ce^{-p/2}(p+2)^{p/2} \quad \forall p \geq 2.$$

Moreover, note that so far $C = \sqrt{2}$ is not guaranteed, yet from our Theorem 1.1 below it follows that in fact Egorov's bound (1.3) is valid for all real $p \geq 2$ with minimal constant $C = \sqrt{2}$, which in turn gives a very good approximation of the optimal value, as we will see in Lemma 1.3.

Theorem 1.1. *Let Γ denote the Gamma function and let p_0 be the solution of the equation $\Gamma\left(\frac{p+1}{2}\right) = \sqrt{\pi}/2$ in the interval $(1, 2)$, i.e. $p_0 \approx 1.84742$. Then for every $p > 0$ it holds:*

$$A_{p,opt} = A_{p,H} := \begin{cases} 2^{p/2-1}, & 0 < p \leq p_0 \\ 2^{p/2}\,\dfrac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}, & p_0 < p < 2 \\ 1, & 2 \leq p < \infty \end{cases}$$

and

$$B_{p,opt} = B_{p,H} := \begin{cases} 1, & 0 < p \leq 2 \\ 2^{p/2}\,\dfrac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}, & 2 < p < \infty. \end{cases}$$

Consider the special case that $p = 2k,\ k \in \mathbb{N}$, is an even integer. It is well known that $\Gamma\left(k+\frac{1}{2}\right) = \frac{(2k)!}{k!2^{2k}}\sqrt{\pi}$, whence

$$B_{2k,opt} = 2^k\,\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}} = \frac{(2k)!}{k!2^k} = (2k-1)!!, \tag{1.4}$$

where the last equality is easily seen by induction. Thus we obtain

Corollary 1.2. *If $X_1, \ldots, X_n$ are independent and symmetric with $\mathbb{E}[X_i^{2k}] < \infty$, $1 \leq i \leq n$ for some $k \in \mathbb{N}$, then*

$$\mathbb{E}\left[\left|\sum_{i=1}^{n} X_i\right|^{2k}\right] \leq (2k-1)!!\;\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i^2\right)^{k}\right].$$

Ferger (in press, Theorem 1.4) gives a direct proof of Corollary 1.2 by using a combinatorial method.

Let

$$B_{p,E} := \sqrt{2}e^{-p/2}p^{p/2}, \quad p \geq 2,$$

be the best possible choice for Egorov's bound. Then $B_{p,opt}$ is strictly smaller than $B_{p,E}$ for all $p \geq 2$ and the corresponding relative errors converge to zero as p tends to infinity. More precisely, we have

Lemma 1.3. $\exp\left\{-\frac{1}{4p}\right\} B_{p,E} < B_{p,opt} < \exp\left\{-\frac{1}{18p}\right\} B_{p,E}$ *for all $p \geq 2$ with relative error*

$$\frac{|B_{p,E} - B_{p,opt}|}{B_{p,opt}} \leq \frac{1}{4p}\, e^{1/(4p)} \to 0, \quad p \to \infty.$$