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A note on likelihood ratio ordering of order statistics from two samples

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ABSTRACT

The purpose of this paper is to establish the stochastic comparisons of order statistics from two samples in the sense of likelihood ratio order. We strengthen and complement some results in Zhao and Balakrishnan (2012) and Ding et al. (2013).

1. Introduction

Order statistics are widely used in reliability, data analysis, goodness-of-fit, statistical inference and other areas. There is a large amount of the literature on stochastic comparisons of order statistics in the past ten years. One may refer to Shaked and Shanthikumar (2007) and references therein. Let $X_1, \ldots, X_p$ be a random sample of size p from a distribution F and $X_{p+1}, \ldots, X_n$ be another independent random sample of size q from a distribution G, where $n = p+q$ and $0 \le p < n$. It is further assumed that the two samples are independent. Denote by $X_{1:n}(p,q) \le X_{2:n}(p,q) \le \cdots \le X_{n:n}(p,q)$ the order statistics of the two samples $X_1, \ldots, X_n$. For stochastic comparisons of order statistics from two samples, we refer the reader to Khaledi and Kochar (2001), Hu and Zhu (2003), Hu et al. (2006), Wen et al. (2007), Kochar and Xu (2011), Zhao and Balakrishnan (2012), and Ding et al. (2013). As special consequences, it follows from Ding et al. (2013) that, for any $1 \le k \le n$ and $0 \le p < n$,

$$G \le_{\rm hr} F \Longrightarrow X_{k:n}(p,q) \le_{\rm hr} X_{k:n}(p+1, q-1), \tag{1.1}$$
$$G \le_{\rm rh} F \Longrightarrow X_{k:n}(p,q) \le_{\rm rh} X_{k:n}(p+1, q-1), \tag{1.2}$$

and

$$G \le_{\rm mrl} F \Longrightarrow X_{n:n}(p,q) \le_{\rm mrl} X_{n:n}(p+1, q-1), \tag{1.3}$$

where $\le_{\rm hr}$, $\le_{\rm rh}$ and $\le_{\rm mrl}$ are the hazard rate, the reversed hazard rate and the mean residual life orders, respectively. The formal definition of the stochastic order is given in Section 2.

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Order statistics have a close connection with the lifetimes of k-out-of-n systems. In reliability engineering, a system consisting of n components is called a k-out-of-n system when it functions if and only if at least k components are functioning. Let B_1 and B_2 be two different batches of components. In each batch, all components have a common distribution function. Consider a k-out-of-n system with p components taken from B_1 and $q = n - p$ components from B_2 and assume that all components function independently. Then, the lifetime $T_{k:n}(p, q)$ of the k-out-of-n system is the same as $X_{n-k+1:}(p, q)$.

The purpose of this short note is to establish the analog of (1.1) and (1.2) for the likelihood ratio order, that is, for any $1 \le k \le n$ and $0 \le p < n$,

$$G \le_{\mathrm{lr}} F \Longrightarrow X_{k:n}(p, q) \le_{\mathrm{lr}} X_{k:n}(p+1, q-1).$$

The main result and its proof are given in Section 3, while the definitions of some stochastic orders and of permanents are recalled in Section 2.

2. Preliminaries

In this section, we first recall some definitions of stochastic orders that will be useful in this paper and then give a representation of the density function of $X_{k:n}(p, q)$ in terms of permanents.

Let $\overline{F}$ and $\overline{G}$ be the survival functions of random variables X and Y, respectively. Then X is said to be smaller than Y in the *usual stochastic order*, denoted by $X \le_{\mathrm{st}} Y$, if $\overline{F}(t) \le \overline{G}(t)$ for all t. Also, X is said to be smaller than Y in the *hazard rate order*, denoted by $X \le_{\mathrm{hr}} Y$, if $\overline{G}(t)/\overline{F}(t)$ is increasing in t for which the ratio is well defined. On the other hand, X is said to be smaller than Y in the *reversed hazard rate order*, denoted by $X \le_{\mathrm{rh}} Y$, if $G(t)/F(t)$ is increasing in t. If f and g are the respective density or mass functions of F and G, and $g(t)/f(t)$ is increasing in t, then X is said to be smaller than Y in the *likelihood ratio order*, denoted by $X \le_{\mathrm{lr}} Y$. The likelihood ratio order implies the hazard rate and reversed hazard rate orders, which in turn imply the usual stochastic order. We also write $F \le_* G$ if $X \le_* Y$, where $\le_*$ is $\le_{\mathrm{st}}$, $\le_{\mathrm{hr}}$, $\le_{\mathrm{rh}}$ or $\le_{\mathrm{lr}}$. For more on stochastic orders, see Shaked and Shanthikumar (2007).

It is useful to represent the (joint) density functions of order statistics by using the theory of permanents when the underlying random variables are not identical (see Bapat and Kochar, 1994, Hu et al., 2001, and Hu and Zhu, 2003). If $\mathbf{A} = (a_{ij})$ is an $n \times n$ matrix, then the *permanent* of $\mathbf{A}$ is defined as

$$\mathrm{perm}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation is over all permutations $\sigma = (\sigma(1), \ldots, \sigma(n))$ of $\{1, \ldots, n\}$. If $\mathbf{d}_1, \ldots, \mathbf{d}_n$ are vectors in $\Re^n$, then we will denote by $[\mathbf{d}_1, \ldots, \mathbf{d}_n]$ the permanent of the $n \times n$ matrix $(\mathbf{d}_1, \ldots, \mathbf{d}_n)$. The permanent

$$\Bigg[\underbrace{\mathbf{d}_1}_{r_1}, \underbrace{\mathbf{d}_2}_{r_2}, \ldots \Bigg]$$

is obtained by taking r_1 copies of $\mathbf{d}_1$, r_2 copies of $\mathbf{d}_2$, and so on. Denote by f and g the density functions of F and G, respectively, and let $\mathbf{e}_m$ be the column vector of size m of all ones. For each pair (p, q), define

$$\mathbf{F}_{p,q}(x) = \begin{pmatrix} F(x)\mathbf{e}_p \\ G(x)\mathbf{e}_q \end{pmatrix}, \qquad \overline{\mathbf{F}}_{p,q}(x) = \begin{pmatrix} \overline{F}(x)\mathbf{e}_p \\ \overline{G}(x)\mathbf{e}_q \end{pmatrix}, \qquad \mathbf{f}_{p,q}(x) = \begin{pmatrix} f(x)\mathbf{e}_p \\ g(x)\mathbf{e}_q \end{pmatrix}.$$

Then, for $k = 1, \ldots, n$, the density function of $T_{k|n}(p, q) = X_{n-k+1:n}(p, q)$ is given by

$$f_{T_{k|n}}(x, p) = \frac{1}{(n-k)!(k-1)!} \Bigg[\underbrace{\mathbf{F}_{p,q}(x)}_{n-k} \quad \underbrace{\mathbf{f}_{p,q}(x)}_{1} \quad \underbrace{\overline{\mathbf{F}}_{p,q}(x)}_{k-1} \Bigg], \quad x \in \Re. \tag{2.1}$$

The following lemma is the well-known Alexandroff inequality for permanents.

Lemma 2.1 (Van Lint, 1981)**.** *Let $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}$ be non-negative vectors in $\Re^n$, and let $\mathbf{b}$ be any vector in $\Re^n$, where $n \ge 2$. Then*

$$[\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{b}]^2 \ge [\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{a}_{n-1}] \cdot [\mathbf{a}_1, \ldots, \mathbf{a}_{n-2}, \mathbf{b}, \mathbf{b}].$$

3. The main result

The main result of this paper is as follows.

Theorem 3.1. *Let $X_1, \ldots, X_p$ be a random sample of size p from a distribution F and $X_{p+1}, \ldots, X_n$ be another independent random sample of size q from a distribution G, where $n = p + q$ and $0 \le p < n$. Then, for $k = 1, \ldots, n$,*

$$G \le_{\mathrm{lr}} F \Longrightarrow X_{k:n}(p, q) \le_{\mathrm{lr}} X_{k:n}(p+1, q-1). \tag{3.1}$$