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# A note on Monge–Kantorovich problem

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ABSTRACT

Shen and Zheng (2010) and Xu and Yan (2013) considered the Monge–Kantorovich problem in the plane and proved that the optimal coupling for the problem has a form $(X_1, g(X_1, Y_2), h(X_1, Y_2), Y_2)$, and then they assumed $(X_1, Y_2)$ has a density $p$ and gave the equation which $p$ should satisfy. In this article, we prove that $(X_1, Y_2)$ naturally has a density under more weak conditions. We again prove a similar result in dimension 3 and give an exact form $(X_1, g_1(X_1, Y_2, Y_3), g_2(X_1, Y_2, Y_3), h(X_1, Y_2, Y_3), Y_2, Y_3)$ depending on a certain convex function.



## 1. Introduction and notations

Let $\mathcal{L}(\mathcal{F}, \mathcal{G})$ be the set of all $2n$-dimensional random variables whose marginal distribution functions are $\mathcal{F}$ and $\mathcal{G}$, respectively. The so-called Monge–Kantorovich problem is to find an optimal coupling of $(X, Y) \in \mathcal{L}(\mathcal{F}, \mathcal{G})$ such that $\mathbb{E}|X-Y|^2$ attains the minimum.

In this article, we assume that $\mathcal{F}$ has a density $f$ that means $\mathcal{F}(x) = \int_{-\infty}^{x} f(x')dx'$, $\mathcal{G}$ has a density $\tilde{f}$. $\mu$, $\nu$ are two measures on $\mathbb{R}^n$ such that $\mu(dx) = f(x)dx$, $\nu(dx) = \tilde{f}(x)dx$.

In dimension 2 ($n = 2$), Shen and Zheng (2010) and Xu and Yan (2013) proved that the optimal coupling of $\mathcal{F}$ and $\mathcal{G}$ has the following form:

$$(X_1, g(X_1, Y_2), h(X_1, Y_2), Y_2);$$

here $g, h$ are some functions depending on $f, \tilde{f}$ and the law of $(X_1, Y_2)$. Then, Shen and Zheng (2010) and Xu and Yan (2013) assumed that $Z = (X_1, Y_2)$ has a density and gave the equation which $p(.,.)$ should satisfy.

In this article, in Section 2, we consider the situation with dimension 2 ($n = 2$) and prove that if

$$(X_1, X_2, Y_1, Y_2),$$

is the optimal coupling of $\mathcal{F}$ and $\mathcal{G}$, then the law of $(X_1, Y_2)$ is naturally absolutely continuous with the Lebesgue measure on $\mathbb{R}^2$.

In Section 3, we consider the situation with dimension 3 ($n = 3$). First, we prove that if $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ is the optimal coupling of $\mathcal{F}$ and $\mathcal{G}$, then $(X_1, Y_2, Y_3)$ has a density naturally. Then we prove that the optimal coupling of $\mathcal{F}$ and $\mathcal{G}$ can be assumed to have the following form:

$$(X_1, g_1(X_1, Y_2, Y_3), g_2(X_1, Y_2, Y_3), h(X_1, Y_2, Y_3), Y_2, Y_3); \tag{1}$$

here the functions $g_1, g_2, h$ depend on $f, \tilde{f}$ and the law of $(X_1, Y_2, Y_3)$. Let $p$ be the density of $(X_1, Y_2, Y_3)$, and then we give the expression of $p$ in some sense.

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In the following particular Monge–Ampère equation,

$$F(x) = \det(D^2\varphi(x)), \tag{2}$$

$F(x)$ plays simultaneously the role of right hand side and coefficients due to the structure of its nonlinearity. The standard argument shows that this equation is elliptic only when $D^2\varphi(x)$ is a positive definite matrix, equivalently, (2) is elliptic only for functions $\varphi$ that is strictly convex in its domain. To ensure (2) exists a solution, $F(x)$ must be positive. Let $C^\alpha$ be the collection of functions which are Hölder continuous of order $\alpha$. In this article, we make the following assumptions on $f$ and $\tilde{f}$:

**(H)** $f, \tilde{f} \in C^\alpha$, $\alpha \in (0, 1)$ and for any $x$, $f(x)$ and $\tilde{f}(x)$ are positive.

Hence we do not need $f; \tilde{f}$ are as smooth as that in Xu and Yan (2013). From Theorem 11 in Villani (2002), there exists unique mappings $\nabla\varphi$ and $\nabla\varphi^*$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $\nabla\varphi\#\mu = \nu$ and $\nabla\varphi^*\#\nu = \mu$, here $\varphi$ and $\varphi^*$ are convex functions from $\mathbb{R}^n$ to $\mathbb{R}$. Then $\varphi$ and $\varphi^*$ are convex functions satisfying the following specific Monge–Ampère equations:

$$f(x) = \tilde{f}(\nabla\varphi(x)) \det(D^2\varphi(x)), \quad x \in \mathbb{R}^n \tag{3}$$

$$\tilde{f}(x) = f(\nabla\varphi^*(x)) \det(D^2\varphi^*(x)), \quad x \in \mathbb{R}^n. \tag{4}$$

Similar to the result in linear elliptic equations, it can be obtained that $\varphi \in C^{2,\alpha}$ by using standard but more complicated continuity methods.

Furthermore, by bootstrapping argument, there is the following proposition.

**Proposition 1.1** (*Cf. Caffarelli (2002)*). *If $f$ and $\tilde{f}$ never vanish or if the supports of $f$ and $\tilde{f}$ are convex, then the regularity of $\nabla\varphi(x)$ is "one derivative better" than $f$ and $\tilde{f}$.*

**Remark 1.1.** The solution is not unique for the general Monge–Ampère equation in a full space since it has a very rich family of invariants. But for our particular case, $\det(D^2\varphi(x)) = \frac{f(x)}{\tilde{f}(\nabla\varphi(x))}$, because of the special structure on the right hand side, the uniqueness of $\nabla\varphi(x)$ can be proved (cf. Villani, 2002, Theorem 11); here we require $\varphi(x)$ to be a convex function.

For any function $\phi : \mathbb{R}^n \to \mathbb{R}$, we denote by $\nabla_1\phi = \nabla_{x_1}\phi(x_1, \ldots, x_n), \ldots, \nabla_n\phi = \nabla_{x_n}\phi(x_1, \ldots, x_n)$, $\nabla\phi = (\nabla_1\phi, \ldots, \nabla_n\phi)$. If $X, Y$ are two random variables, $X \sim Y$ means $X$ and $Y$ have the same distribution. If $\mu_0$ is a measure, then $X \sim \mu_0$ means the distribution function of $X$ is given by $F(x) = \mu_0((-\infty, x])$.

## 2. Case with dimension 2

When $n = 2$ (in the plane), Shen and Zheng (2010) and Xu and Yan (2013) proved that the optimal coupling of $\mathcal{F}$ and $\mathcal{G}$ has the following form:

$$(X_1, g(X_1, Y_2), h(X_1, Y_2), Y_2),$$

for some functions $g, h$ depending on $f, \tilde{f}$ and the law of $(X_1, Y_2)$. They assumed that the 2-dimensional random vector $\mathcal{Z} = (X_1, Y_2)$ has a density $p(., .)$ and gave the equation $p$ should satisfy.

In this section, if

$$(X_1, X_2, Y_1, Y_2),$$

is the optimal coupling of $\mathcal{F}$ and $\mathcal{G}$, we prove that the law of $(X_1, Y_2)$ is naturally absolutely continuous with the Lebesgue measure on $\mathbb{R}^2$. Define mapping

$$Q : (x_1, x_2) \to (Q_1(x_1, x_2), Q_2(x_1, x_2)) = (x_1, \nabla_2\varphi(x_1, x_2))$$

and $\mathrm{Range}(Q) := \{(x_1, \nabla_2\varphi(x_1, x_2)) : (x_1, x_2) \in \mathbb{R}^2\}$. To prove the main theorem in this section, we need the following two lemmas.

**Lemma 2.1.** *If $E \subseteq \mathbb{R}^m$ is a measurable set, $T : \mathbb{R}^m \to \mathbb{R}^m$, if*

(1) *$T^{-1} : T(E) \to E$ exists,*
(2) *$T$ and $T^{-1}$ map the measurable set to measurable set,*

*then there exists a integrable function $J_T$ such that*

$$\int_{T(E)} f(x)dx = \int_E f(Ty)J_T(y)dy, \quad \forall f \in L^1(T(E)).$$

**Lemma 2.2.** *The mapping $Q$ is injective.*

**Proof.** From (3),

$$\det(D^2\varphi(x)) = \nabla_{22}\varphi(x)\nabla_{11}\varphi(x) - (\nabla_{12}\varphi)^2 = f(x)/\tilde{f}(\nabla\varphi(x)) > 0.$$