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The closure of the convolution equivalent distribution class under convolution roots with applications to random sums[☆]

Changjun Yu[a], Yuebao Wang[a,*], Yang Yang[a,b]

[a] Department of Mathematics, Soochow University, Suzhou 215006, PR China

[b] Department of Applied Mathematics, Nanjing Audit University, Nanjing 210029, PR China

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ABSTRACT

Let F be a proper distribution on $D = [0,\infty)$ or $(-\infty,\infty)$ and N be a non-negative integer-valued random variable with masses $p_n = P(N = n)$, $n \geq 0$. Denote $G = \sum_{n=0}^{\infty} p_n F^{*n}$. The main result of this paper is that under some suitable conditions, G belongs to the convolution equivalent distribution class if and only if F belongs to the convolution equivalent distribution class. As applications, some known results on random sums have been extended and improved, which give a positive answer under certain conditions to Problem 1 of Watanabe (2008). Similarly, some corresponding results for the local distributions and densities have been obtained.

1. Introduction

It is well known that the closure under convolutions and convolution roots of some distribution classes has a wide range of important applications in queueing theory, risk theory, branching process theory, infinite divisibility theory and so on. Compared with the development of the closure under convolutions, there have been long and difficult studies undertaken on the closure under convolution roots. A main problem that has not been completely solved is the closure under convolution roots under the condition (1.2) below. This paper aims to discuss this problem. In order to better illuminate our motivation and results, we first introduce some notions and notation, which will be valid in the rest of this paper.

Unless otherwise stated, in this paper a limit is taken as $x \to \infty$. Let $a(x)$ and $b(x)$ be non-negative functions on D, where $D = (-\infty,\infty)$ or $[0,\infty)$. Define $a \otimes a(x) = a^{\otimes 2}(x) = \int_D a(x-y)a(y)\mathrm{d}y$. We write $a(x) \sim b(x)$, if $\lim a(x)/b(x) = 1$; $a(x) = O(b(x))$, if $\limsup a(x)/b(x) < \infty$; $a(x) \approx b(x)$, if $a(x) = O(b(x))$ and $b(x) = O(a(x))$; $a(x) = o(b(x))$, if $\lim a(x)/b(x) = 0$. We say that $a \in \mathcal{L}d(\gamma)$ for some $\gamma \geq 0$, if $a(x) > 0$ for sufficiently large x and $a(x-t) \sim \mathrm{e}^{\gamma t} a(x)$ for any $t \in (-\infty,\infty)$, and say that $a \in \mathcal{S}d(\gamma)$ for some $\gamma \geq 0$, if $a \in \mathcal{L}d(\gamma)$, $\int_D \mathrm{e}^{\gamma y} a(y)\mathrm{d}y < \infty$ and $a^{\otimes 2}(x) \sim 2a(x)\int_D \mathrm{e}^{\gamma y} a(y)\mathrm{d}y$.

Let F be a proper distribution on D, i.e. $F(\infty) = 1$. Denote the tail of distribution F by $\overline{F} = 1 - F$. For two distributions F_1 and F_2, denote the convolution of F_1 and F_2 by $F_1 * F_2$, and denote the n-fold convolution of F by F^{*n}, $n = 0, 1, 2, \ldots$, where $F^{*1} = F$ and F^{*0} is the distribution degenerate at zero. We say that $F \in \mathcal{L}(\gamma)$ for some $\gamma \geq 0$, if $\overline{F} \in \mathcal{L}d(\gamma)$, and say that $F \in \mathcal{S}(\gamma)$ for some $\gamma \geq 0$, if $m_F(\gamma) = \int_D \mathrm{e}^{\gamma y} F(\mathrm{d}y) < \infty$, $F \in \mathcal{L}(\gamma)$ and $\overline{F^{*2}}(x) \sim 2m_F(\gamma)\overline{F}(x)$. The class $\mathcal{S}(\gamma)$, $\gamma \geq 0$ is the so-called convolution equivalent distribution class. Especially, we call $\mathcal{S}(0)$ and $\mathcal{L}(0)$ the subexponential distribution class and the long-tailed distribution class, denoted by $\mathcal{S}$ and $\mathcal{L}$, respectively. These distribution classes were first introduced by Chistyakov (1964) for $\gamma = 0$, and by Chover et al. (1973a,b) for $\gamma > 0$. Bertoin and Doney (1996) pointed out that in the definition of $\mathcal{L}(\gamma)$, when $\gamma > 0$ and F is a lattice distribution, x and y should be taken to be an integer multiple of the span,

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* Corresponding author. Tel.: +86 512 67422726; fax: +86 512 65112637.
E-mail address: ybwang@suda.edu.cn (Y. Wang).

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which we will assume to be the case in the following. Klüppelberg (1988) introduced an important subclass of $\mathcal{S}$, denoted by $\mathcal{S}^*$. Say that $F \in \mathcal{S}^*$, if $\overline{F} \in \mathcal{S}d(0)$. In addition, Corollary 2.2 of Klüppelberg (1989) proved that for $\gamma > 0$, $F \in \mathcal{S}(\gamma)$ if and only if $\overline{F} \in \mathcal{S}d(\gamma)$.

Now we turn to the closure under convolution roots. Let N be a non-negative integer-valued random variable (r.v.) with masses $p_n = P(N = n)$, $n \geq 0$. In this paper, we always assume that there exists some integer $n \geq 1$ such that $p_n > 0$. Denote $G = \sum_{n=0}^{\infty} p_n F^{*n}$. For detailed studies to G, see Section 2.5 of Embrechts et al. (1997) and references therein. The problem about the closure of $\mathcal{S}(\gamma)$, $\gamma \geq 0$ under convolution roots is that, under which conditions we can get the following assertion

$$G \in \mathcal{S}(\gamma) \Rightarrow F \in \mathcal{S}(\gamma). \tag{1.1}$$

We first briefly review the history of the research on the closure of $\mathcal{S}(\gamma)$, $\gamma \geq 0$ under convolution roots. If $N \equiv n \geq 2$ and $F \in \mathcal{L}(\gamma)$ for some $\gamma \geq 0$, Theorem 2.10 of Embrechts and Goldie (1982) proved (1.1) for the case that $D = [0, \infty)$, and Theorem 5.1 of Pakes (2007) proved (1.1) for the case that $D = (-\infty, \infty)$. If $N \equiv n \geq 2$ and $\gamma = 0$, Theorem 2 of Embrechts et al. (1979) proved (1.1) for the case that $D = [0, \infty)$, and Proposition 2.7(ii) of Watanabe (2008) proved (1.1) for the case that $D = (-\infty, \infty)$. If N is a Poisson r.v., Theorem 3 of Embrechts et al. (1979) proved (1.1) for the case that $\gamma = 0$ and $D = [0, \infty)$. For the case that $\gamma > 0$ and $D = [0, \infty)$, Theorem 4.2 of Embrechts and Goldie (1982) obtained (1.1) under a technical condition. Theorem 3.1 of Pakes (2004) proved (1.1) for the case that $\gamma = 0$ and $D = (-\infty, \infty)$. Theorem 1.1 of Watanabe (2008) proved (1.1) in the case that $\gamma > 0$ and $D = (-\infty, \infty)$. For a more general r.v. N, Theorem 2.13 of Cline (1987) proved (1.1) for the case that $F \in \mathcal{L}(\gamma)$ for some $\gamma \geq 0$ and $D = [0, \infty)$. Theorem 5.1 of Pakes (2004) extended Cline's result to the case that $D = (-\infty, \infty)$. The main condition used by the last two mentioned results is that

$$\sum_{n=0}^{\infty} p_n((m_F(\gamma) + \varepsilon) \vee 1)^n < \infty \quad \text{for some } \varepsilon > 0. \tag{1.2}$$

Obviously, if $m_F(\gamma) < 1$, (1.2) is naturally satisfied by each r.v. N. If $\gamma = 0$, (1.2) holds for every light-tailed r.v. N, i.e. $Ee^{sN} < \infty$ for some $s > 0$. However, as stated in Remark 4.2 of Shimura and Watanabe (2005), the last two results are based on Lemma 2.1(iv) of Cline (1987), which is incorrect. Thus they should be reconsidered.

Recently, (1.1) has been shown by some papers under some stronger conditions without using Lemma 2.1(iv) of Cline (1987). For $D = (-\infty, \infty)$ and $F \in \mathcal{L}(\gamma)$, both Pakes (2007) and Wang et al. (2007) proved (1.1) under the condition that $\overline{G}(x) = O(\overline{F}(x))$. Further, for $D = [0, \infty)$ and $F \in \mathcal{L}(\gamma)$, Lemma 3.4 of Pakes (2007) proved (1.1) under the following condition that

$$\sum_{n=1}^{\infty} p_n(q^{-1}(m_G(\gamma) + \varepsilon))^n < \infty \quad \text{for some } \varepsilon > 0, \tag{1.3}$$

where $q = \sum_{n=1}^{\infty} p_n$. For $D = (-\infty, \infty)$ and $F \in \mathcal{L}(\gamma)$, (1.1) was obtained respectively by Theorem 1.2 of Wang et al. (2007) under the condition that

$$\limsup_{n\to\infty} p_{n+1}/p_n < (m_F(\gamma))^{-1} \tag{1.4}$$

and by Proposition 1.7 of Watanabe (2008) under the condition that

$$\sum_{n=0}^{\infty} p_n((r^{-1} m_G(\gamma) + \varepsilon) \vee 1)^n < \infty \quad \text{for some } \varepsilon > 0, \tag{1.5}$$

where $r = \sum_{n=1}^{\infty} p_n(F[0, \infty))^{n-1}$.

However, the above conditions (1.3)–(1.5) are stronger than (1.2). Thus whether Theorem 5.1 of Pakes (2004) is right or wrong is unknown. In this paper, we will show that Theorem 5.1 of Pakes (2004) still holds, which gives a positive answer under the conditions (1.2) and $F \in \mathcal{L}(\gamma)$ to Problem 1 in Watanabe (2008, p. 371). Thus, Theorem 1.2 of Wang et al. (2007), Lemma 3.4 of Pakes (2007) and Proposition 1.7 of Watanabe (2008) have been improved. Using this result, Theorem 4.1 of Watanabe (2008) still holds without the conditions (4.3) and (4.4) of that paper. Now we give the main result of this paper.

Theorem 1.1. *Let F be a distribution on $D = (-\infty, \infty)$. When F is light-tailed, assume that $F \in \mathcal{L}(\gamma)$ for some $\gamma > 0$ and $m_F(\gamma) < \infty$. Also let the masses p_n, $n \geq 0$ of the r.v. N satisfy (1.2). Then (1.1) holds.*

From Theorem 1.1, we can immediately get the following corollary.

Corollary 1.1. *Let F be a distribution on D. Suppose that $F \in \mathcal{L}(\gamma)$ for some $\gamma \geq 0$ and (1.2) holds, then the following assertions are equivalent:*

(a) $F \in \mathcal{S}(\gamma)$;
(b) $\overline{G}(x) \sim \sum_{n=1}^{\infty} n p_n (m_F(\gamma))^{n-1} \overline{F}(x)$;
(c) $G \in \mathcal{S}(\gamma)$.