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Sufficient conditions for stochastic equality of two distributions under some partial orders

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ABSTRACT

We establish some conditions for stochastic equality of two nonnegative random variables which are ordered with respect to variability ordering or with respect to mean residual life ordering or with respect to second order stochastic ordering.

1. Introduction

Let X and Y be two nonnegative random variables with distribution functions F and G respectively. Let $\bar{F}$ and $\bar{G}$ denote the corresponding survival functions.

Definition 1.1. The random variable X is said to be *stochastically* larger than the random variable Y (denoted by $X \overset{st}{\geq} Y$) if

$$\bar{F}(x) \geq \bar{G}(x) \quad \text{for every } x \geq 0$$

(cf. Shaked and Shantikumar (2007)).

Definition 1.2. The random variable X is said to be larger than the random variable Y in *variability ordering* (denoted by $X \overset{v}{\geq} Y$) if

$$\int_x^\infty \bar{F}(u)\mathrm{d}u \geq \int_x^\infty \bar{G}(u)\mathrm{d}u \quad \text{for every } x \geq 0$$

(cf. Ross (1983)).

Definition 1.3. The random variable X is said to be larger than the random variable Y in *second order stochastic ordering* (denoted by $X \overset{ssd}{\geq} Y$) if

$$\int_0^x \bar{F}(u)\mathrm{d}u \geq \int_0^x \bar{G}(u)\mathrm{d}u \quad \text{for every } x \geq 0$$

(cf. Stoyan (1983), Deshpande and Singh (1985) and Kaur et al. (1994)).

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Definition 1.4. The random variable X is said to be larger than the random variable Y in *mean residual life ordering* (denoted by $X \overset{mrl}{\geq} Y$) if

$$\frac{1}{\bar{F}(x)}\int_x^\infty \bar{F}(u)\mathrm{d}u \geq \frac{1}{\bar{G}(x)}\int_x^\infty \bar{G}(u)\mathrm{d}u \quad \text{for every } x \geq 0$$

(cf. Singh (1989) and Shaked and Shantikumar (2007)).

Bacelli and Makowski (1989) proved the following theorem giving sufficient conditions for stochastic equality of the random variables X and Y when these are known to be stochastically ordered.

Theorem 1.1. *Suppose that $X \overset{st}{\geq} Y$ and $E(h(X)) = E(h(Y))$ for some strictly increasing function $h(.)$. Then $X \overset{st}{=} Y$, that is, the random variables X and Y have the same distribution.*

Bacelli and Makowski (1989) have extended the result stated in Theorem 1.1 to random vectors. Scarsini and Shaked (1990) established some sufficient conditions for stochastic equality of two random vectors (cf. Theorem 1.A.8, Theorem 6.B.19, Shaked and Shantikumar (2007)).

Our aim in this paper is to obtain sufficient conditions for stochastic equality of two nonnegative random variables which are ordered with respect to the partial orders defined in Definitions 1.2–1.4.

2. Some conditions for stochastic equality

We now prove a result similar to Theorem 1.1 for nonnegative random variables X and Y ordered with respect to the variability ordering.

Theorem 2.1. *Suppose X and Y are nonnegative random variables such that $X \overset{v}{\geq} Y$ and $E(h^2(X)) = E(h^2(Y))$ for some strictly increasing convex positive function $h(.)$ Then the random variables X and Y have the same distribution.*

Proof. Note that $X \overset{v}{\geq} Y$ if

$$\int_x^\infty \bar{F}(u)\mathrm{d}u \geq \int_x^\infty \bar{G}(u)\mathrm{d}u \quad \text{for every } x \geq 0$$

by definition which holds if and only if

$$E[\phi(X)] \geq E[\phi(Y)]$$

for all increasing convex functions $\phi(.)$ (cf. Ross (1983)). Let $Z = h(X)$ and $W = h(Y)$. For any increasing convex function $\phi(.)$, the function $\phi(h(x))$ is clearly increasing and convex implying that

$$E[\phi(h(X))] \geq E[\phi(h(Y))]$$

which in turn implies that

$$Z \overset{v}{\geq} W.$$

Let $\bar{H}$ and $\bar{K}$ denote the survival functions of the random variables Z and W respectively. Then

$$\int_x^\infty \bar{H}(u)\mathrm{d}u \geq \int_x^\infty \bar{K}(u)\mathrm{d}u \quad \text{for every } x \geq 0.$$

Let

$$L(x) = \int_x^\infty [\bar{H}(u) - \bar{K}(u)]\mathrm{d}u.$$

Note that $L(x) \geq 0$ for all $x \geq 0$. We will now prove that $L(x) = 0$ for all $x \geq 0$. On the contrary, suppose $L(x) > 0$ for some $x = x_0 \geq 0$. Since the function $L(.)$ is continuous on the interval $[0, \infty)$, it follows that there exists a neighbourhood of x_0 in which the function $L(.)$ is strictly positive. Hence

$$\int_0^\infty L(x)\mathrm{d}x > 0$$

which implies that

$$\int_0^\infty \int_x^\infty \bar{H}(u)\mathrm{d}u\mathrm{d}x > \int_0^\infty \int_x^\infty \bar{K}(u)\mathrm{d}u\mathrm{d}x.$$