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On the second-order correlation of characteristic polynomials of Hermite β ensembles

Zhonggen Su *

Department of Mathematics, Zhejiang University, 310027, PR China

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ABSTRACT

Consider the Hermite β ensemble, a variant of the classical Gaussian unitary ensemble. Using Dumitriu and Edelman's matrix model representation, we first calculate the generating function of the second-order correlation of characteristic polynomials. Then we obtain the asymptotic behaviors of the second-order correlation of characteristic polynomials both in the bulk ($0 < \beta < 4$) and at the edge ($\beta > 0$). Analogs have recently been studied by Götze and Kösters for general Hermitian (real) Wigner matrices.

1. Introduction and main results

There have been a number of important advances in random matrix theory in the past few decades. One of them is the discovery of the Tracy–Widom law, which describes the limiting distribution of the extreme eigenvalues of Hermitian matrix ensembles and the deep relations between these and the Painlevé functions. The other is the striking link discovered in Keating and Snaith (2000) between the asymptotic moments of the characteristic polynomial of a random matrix from the Circular Unitary Ensemble (CUE) and the asymptotic moments of the value distribution of the Riemann-zeta function along the critical line. Though the link has not been proved yet, the characteristic polynomials of random matrices have since then attracted considerable interest. Brézin and Hikami (2000, 2001) considered correlation functions of products of characteristic polynomials for an arbitrary unitary invariant ensemble of Hermitian matrices. This ensemble is normally characterized by the weight $e^{-V(M)}$ in the corresponding probability measure, where $V(x)$ is an essentially arbitrary potential function, and the matrix M has dimension n. In particular, it has the following joint eigenvalue probability density function:

$$p_n(x_1, \dots, x_n) = \frac{1}{Z_{n,V}} \prod_{1\le j<k\le n} (x_j - x_k)^2 \prod_{j=1}^{n} e^{-V(x_j)}, \quad x_1, \dots, x_n \in \mathbb{R}. \tag{1.1}$$

Using the method of orthogonal polynomials, they found both exact and asymptotic (large n) expressions for correlations and for the positive moments. Strahov and Fyodorov (2003) then derived the negative moments of characteristic polynomials via the Deift–Zhou steepest-descent/stationary phase method for Riemann–Hilbert problems. In the special cases of the Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE), i.e., $V(x) = x^2$, Borodin and Strahov (2006) employed a discrete approximation technique to provide explicit

* Tel.: +86 57187953676.
E-mail address: suzhonggen@zju.edu.cn.

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expressions for the averages of ratios of products of characteristic polynomials in terms of those that involve only one or two determinants. Only recently did Krasovsky (2008) obtain large n asymptotics for products of noninteger powers of the absolute values of the characteristic polynomials in the GUE. We remark that the unitary invariance property of ensembles plays a crucial role in all of these cited articles.

In this paper we are interested in a variant of the GUE, the Hermite β ensemble. This is a family of probability measures of n unordered, real-valued points $\lambda_1, \lambda_2, \ldots, \lambda_n$ on $\mathbb{R}^n$ whose probability density is given by

$$p_{n,\beta}(x_1, \ldots, x_n) = \frac{1}{Z_{n,\beta}} \prod_{1\le j<k\le n} |x_j - x_k|^\beta \prod_{j=1}^{n} e^{-x_j^2/2}, \quad x_1, \ldots, x_n \in \mathbb{R} \tag{1.2}$$

where $Z_{n,\beta}$ is the appropriate normalizer.

Trivially, when $\beta = 1, 2, 4$, it reduces to the classical Gaussian ensembles. The general β ensembles were first put forward by Dyson as early as the 1960s and appear to be connected to a broad spectrum of areas of mathematics and physics, among which are lattice gases, quantum mechanics, and Selberg-type integrals. However, far less was known beyond the above three special cases until the discovery by Dumitriu and Edelman (2002) of a family of matrix models for all values of β. The so-called matrix model is defined as follows.

Let $\beta > 0$, let $a_1, a_2, \ldots, a_n$ be a sequence of independent random variables with common normal distribution $N(0, 2)$, let $b_1, b_2, \ldots, b_{n-1}$ be a sequence of independent random variables and let each b_i obey the $\chi_{i\beta}$ distribution with density function $P(b_i \in dx) = \frac{2^{1-i\beta/2}}{\Gamma(i\beta/2)} x^{i\beta-1} e^{-x^2/2} dx$, $x > 0$. Assume further that all as and bs are mutually independent. Define

$$A_n = \begin{pmatrix} a_n & b_{n-1} & 0 & 0 & 0 \\ b_{n-1} & a_{n-1} & b_{n-2} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & b_2 & a_2 & b_1 \\ 0 & 0 & 0 & b_1 & a_1 \end{pmatrix}, \qquad H_n^\beta = \frac{1}{\sqrt{2}} A_n. \tag{1.3}$$

The main result of Dumitriu and Edelman (2002) states that H_n^β has eigenvalues with (1.2) as their joint probability density function. The matrix representation (1.3) is due to Trotter (1984) in the case $\beta = 1$; it is however new for other $\beta > 0$ and has inspired a lot of recent activities. For instance, Dumitriu (2003) and Dumitriu and Edelman (2006) used the method of moments together with this new matrix model to prove almost sure convergence of the empirical eigenvalue distribution to the Wigner semicircle law. More precisely, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the H_n^β; then

$$\frac{1}{n} \sum_{k=1}^{n} 1_{\{\lambda_k \le \sqrt{\frac{\beta}{2}n}\, x\}} \longrightarrow \int_{-\infty}^{x} \rho(u) du, \quad \text{a.s.}$$

where

$$\rho(u) = \begin{cases} \frac{1}{2\pi}\sqrt{4-u^2}, & |u| \le 2, \\ 0, & |u| > 2. \end{cases}$$

In addition, let $\lambda_{(1)}$ denote the corresponding largest eigenvalue of the H_n^β. Consider the following stochastic Schrödinger operator

$$H^\beta = \frac{d^2}{dx^2} - x - \frac{2}{\sqrt{\beta}} dW(x),$$

where $W(x)$ is a standard Brownian motion, and denote by Λ_1 the largest eigenvalue of the H^β. Ramírez et al. (in press) used the coupling embedding idea to prove

$$\sqrt{\frac{2}{\beta}}\, n^{1/6} \left(\lambda_{(1)} - \sqrt{2\beta n}\right) \xrightarrow{d} \Lambda_1.$$

Note that this convergence provides a new characterization of the Tracy–Widom-type laws for all β. Thus we have seen that global convergence and fluctuation at the edge is now well understood for general Hermite β ensembles.

Only recently was there new progress for the bulk due to Valkó and Virág (2009). They used again the above tridiagonal matrix to derive a three-term linear recursion for eigenvectors, and hence obtained a bulk-type limit for the eigenvalue point process. In this paper, as a further attempt towards understanding the correlation in the bulk, we shall consider the second-order correlation function of the characteristic polynomials below. Define

$$f_n(u, v) = E \det(uI_n - H_n^\beta) \det(vI_n - H_n^\beta), \tag{1.4}$$

where u, v are real numbers, H_n^β is as in (1.3). We are interested in the asymptotics of the values $f_n(u_n, v_n)$ as $n \to \infty$, where u_n, v_n depend on n in some suitable fashion. The main results are as follows.