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The Brunk–Prokhorov strong law of large numbers for fields of martingale differences taking values in a Banach space

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ABSTRACT

In this paper, we define a new type of fields of martingale differences taking values in Banach spaces and establish the Brunk–Prokhorov strong laws of large numbers and the convergence rate in the strong laws of large numbers for such fields.

1. Introduction and preliminaries

Let $q \geq 1$ and $\{X_n; n \geq 1\}$ be a sequence of independent random variables. The Brunk–Prokhorov strong law of large numbers (Brunk–Prokhorov SLLN) (see Brunk, 1948; Prokhorov, 1950) stated that if $EX_n = 0$, for all $n \geq 1$ and $\sum_{n=1}^{\infty} E|X_n|^{2q}/n^{q+1} < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = 0 \quad \text{a.s.}$$

Brunk–Prokhorov SLLN was extended to martingale differences e.g. in Fazekas and Klesov (2000) and Hu et al. (2008). For the field of random variables with multidimensional index, Lagodowski (2009) established the Brunk–Prokhorov SLLN for fields of independent $\mathbb{E}$-valued random variables and Noszaly and Tomacs (2000) proved the Brunk–Prokhorov SLLN for fields of real-valued martingale differences.

In this paper, we introduce a new type of fields of $\mathbb{E}$-valued martingale differences and establish the Brunk–Prokhorov SLLN for such fields. In Section 1, a new type of fields of $\mathbb{E}$-valued martingale differences is defined, illustrated by some non-trivial examples and compared with the usual definition. In Section 2 we prove some useful lemmas and inequalities. Section 3 contains the main results including the Brunk–Prokhorov SLLN for such fields of $\mathbb{E}$-valued martingale differences.

Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

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Let $\mathbb{E}$ be a real separable Banach space. $(\mathbb{E}, \|\cdot\|)$ is said to be *p-uniformly smooth* $(1 \le p \le 2)$ if there exists a finite positive constant C such that for all $\mathbb{E}$-valued martingales $\{S_n; 1 \le n \le m\}$

$$E\|S_m\|^p \le C \sum_{n=1}^{m} E\|S_n - S_{n-1}\|^p. \tag{1.1}$$

Clearly every real separable Banach space is of 1-uniformly smooth, the real line (the same as any Hilbert space) is of 2-uniformly smooth and the space $L_p(1 \le p \le 2)$ is of p-uniformly smooth. If a real separable Banach space of p-uniformly smooth $(1 < p \le 2)$ then it is of r-uniformly smooth for all $r \in [1, p)$.

Using classical methods from martingale theory, it was shown that (see Woyczyn'ski, 1978) if $\mathbb{E}$ is of p-uniformly smooth, then for all $1 \le q < \infty$ there exists a finite constant C such that

$$E\|S_m\|^q \le CE\left(\sum_{i=1}^{m} \|S_i - S_{i-1}\|^p\right)^{\frac{q}{p}}. \tag{1.2}$$

Let d be a positive integer. For $\mathbf{m} = (m_1, \ldots, m_d)$, $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, denote $\mathbf{m}+\mathbf{n} = (m_1+n_1, \ldots, m_d+n_d)$, $\mathbf{m}-\mathbf{n} = (m_1 - n_1, \ldots, m_d - n_d)$, $|\mathbf{n}| = n_1.n_2 \ldots n_d$, $\|\mathbf{n}\| = \min\{n_1, \ldots, n_d\}$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d$, $\bigvee_{i=1}^{d}(m_i < n_i)$ means that there is at least one of $m_1 < n_1, m_2 < n_2, \ldots, m_d < n_d$ holds. We write $\mathbf{m} \preceq \mathbf{n}$ (or $\mathbf{n} \succeq \mathbf{m}$) if $m_i \le n_i$, $1 \le i \le d$; $\mathbf{m} \prec \mathbf{n}$ if $\mathbf{m} \preceq \mathbf{n}$ and $\mathbf{m} \ne \mathbf{n}$; $\mathbf{m} \ll \mathbf{n}$ (or $\mathbf{n} \gg \mathbf{m}$) if $\bigvee_{i=1}^{d}(m_i < n_i)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathbb{E}$ be a read separable Banach space, and $\mathcal{B}(\mathbb{E})$ be the σ-algebra of all Borel sets in $\mathbb{E}$.

Definition 1. Let $\{X_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ be a field of $\mathbb{E}$-valued random variables and $\{\mathcal{F}_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ be a field of nondecreasing sub-σ-algebras of $\mathcal{F}$ with respect to the partial order $\preceq$ on $\mathbb{N}^d$.

1. The field $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ is said to be an *adapted field* if $X_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable for all $\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}$.
2. The adapted field $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ is said to be a *field of martingale differences in the usual sense* if

$$E(E(X|\mathcal{F}_{\mathbf{m}})|\mathcal{F}_{\mathbf{n}}) = E(X|\mathcal{F}_{\mathbf{m}\wedge\mathbf{n}}) \quad \text{for all } X \in L_1 \tag{1}$$

and

$$E(X_{\mathbf{n}}|\mathcal{F}_{\mathbf{n}}) = 0 \quad \text{for all } \mathbf{1} \preceq n \preceq N \tag{2}$$

(See Christofides and Serfling, 1990; Lagodowski, 2009)

3. The adapted field $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ is said to be a *field of martingale differences* if

$$E(X_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n}}) = 0 \quad \text{for all } \mathbf{1} \preceq n \preceq N \tag{3}$$

where $\mathcal{F}^*_{\mathbf{n}} = \sigma\{\mathcal{F}_{\mathbf{l}} : \vee_{i=1}^{d}(l_i < n_i)\}$, for $\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}$ (see Son et al., 2012).

Remark. For a field of martingale differences the condition (1) about $\{\mathcal{F}_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ is not required but the condition (3) seems to be stronger than the condition (2).

Example 1. Let $\{X_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ be a field of independent random variables with mean 0. Put $\mathcal{F}_{\mathbf{n}} = \sigma(X_{\mathbf{k}}, \mathbf{k} \preceq \mathbf{n})$, then $E(X_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n}}) = 0$ and $\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}$. Therefore, $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ is a field of martingale differences.

Example 2. Let $\{X_n, \mathcal{G}_n : n \ge 1\}$ is a sequences of martingale differences, set

$$X_{\mathbf{n}} = X_n \quad \text{if } \mathbf{n} = (n, n, \ldots, n) \quad \text{and} \quad X_{\mathbf{n}} = 0 \quad \text{if } \mathbf{n} \ne (n, n, \ldots, n);$$
$$\mathcal{G}_{\mathbf{n}} = \mathcal{G}_n \quad \text{if } \mathbf{n} = (n, n, \ldots, n) \quad \text{and} \quad \mathcal{G}_{\mathbf{n}} = \{\emptyset, \Omega\} \quad \text{if } \mathbf{n} \ne (n, n, \ldots, n).$$

Let $\mathcal{F}_{\mathbf{n}} = \sigma\{\mathcal{G}_{\mathbf{k}}, \mathbf{k} \preceq \mathbf{n}\}$ for all $\mathbf{n} \succeq \mathbf{1}$, then $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}} : \mathbf{n} \succeq \mathbf{1}\}$ is a field of martingale differences, but it is not a field of independent random variables.

Example 3. Let $\{X_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ be a field of independent random variables with mean 0. Put $\mathcal{F}_{\mathbf{n}} = \sigma(X_{\mathbf{k}}, \mathbf{k} \preceq \mathbf{n})$ and $Y_{\mathbf{n}} = \prod_{k \prec n} X_{\mathbf{k}}$, if $EY_{\mathbf{n}} < \infty$ for all $\mathbf{n} \preceq \mathbf{N}$, then $E(Y_{\mathbf{n}}|\mathcal{F}^*_{\mathbf{n}}) = 0$ and $\mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}$. Therefore, $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$ is a field of martingale differences, but it is not a field of independent random variables. At the same time, it is not a field of martingale differences in the usual sense since the condition (1) does not hold for $\{\mathcal{F}_{\mathbf{n}}, \mathbf{1} \preceq \mathbf{n} \preceq \mathbf{N}\}$.

Definition 2. Let $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a field of elements in $\mathbb{E}$

1. We say that $a_{\mathbf{n}} \to a$ as $\mathbf{n} \to \infty$ if for any $\epsilon > 0$ there exists $\mathbf{n}_\epsilon \in \mathbb{N}^d$ such that for all $\mathbf{n} \succeq \mathbf{n}_\epsilon$ then $\|a_{\mathbf{n}} - a\| < \epsilon$.
2. We say that $a_{\mathbf{n}} \to a$ strongly as $\mathbf{n} \to \infty$ if for any $\epsilon > 0$ there exists $\mathbf{n}_\epsilon \in \mathbb{N}^d$ such that for all $\mathbf{n} \not\preceq \mathbf{n}_\epsilon$ then $\|a_{\mathbf{n}} - a\| < \epsilon$ (See Lagodowski, 2009).

Clearly, $a_{\mathbf{n}} \to a$ strongly as $\mathbf{n} \to \infty$ then $a_{\mathbf{n}} \to a$ as $\mathbf{n} \to \infty$, but the converse is not true. For example, let $a \ne b$, $a_{(n_1,1\ldots,1)} = b$ and $a_{\mathbf{n}} = a$ if otherwise, then $a_{\mathbf{n}} \to a$ but $a_{\mathbf{n}} \not\to$ strongly as $\mathbf{n} \to \infty$.

It is easy to see that in the case $d = 1$, the strong convergence and the convergence are equivalent.