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Hitting time distribution for skip-free Markov chains: A simple proof[☆]

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ABSTRACT

A well-known theorem for an irreducible skip-free Markov chain on the nonnegative integers with absorbing state d, under some conditions, is that the hitting (absorbing) time of state d starting from state 0 is distributed as the sum of d independent geometric (or exponential) random variables. The purpose of this paper is to present a direct and simple proof of the theorem in the cases of both discrete and continuous time skip-free Markov chains. Our proof is to calculate directly the generation functions (or Laplace transforms) of hitting times in terms of the iteration method.

1. Introduction

The skip-free Markov chain on $\mathbb{Z}^+$ is a process for which upward jumps may be only of unit size, and there is no restriction on downward jumps. Consider a chain starts at 0, and we suppose d is an absorbing state. An interesting property for the chain is that the hitting time of state d when departing from state 0 is distributed as a sum of d independent geometric (or exponential) random variables.

There are many authors who gave different proofs to the results. For the birth and death chain, the well-known results can be traced back to Karlin and McGregor (1959) and Keilson (1971, 1979). Kent and Longford (1983) proved the result for the discrete time version (nearest random walk) although they have not specified the result as a usual form. Fill (2009a) gave the first stochastic proof to both nearest random walk and birth and death chain cases via duality which was established in the paper of Diaconis and Fill (1990). Diaconis and Miclo (2009) presented another probabilistic proof for the birth and death chain. Very recently, Gong et al. (2012) gave a similar result in the case that the state space is $\mathbb{Z}^+$, and they use the well established result to determine all the eigenvalues or the spectrum of the generator.

For the skip-free chain, Brown and Shao (1987) first proved the result in the continuous time situation. By using the duality, Fill (2009b) gave a stochastic proof to both discrete and continuous time cases. The purpose of this paper is to

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present a direct and simple proof of the theorem in the cases of both discrete and continuous time skip-free Markov chains. Our approach is to calculate directly the generation functions (or Laplace transforms) of hitting times in terms of the iteration method.

Theorem 1.1. *For the discrete-time skip-free random walk:*

Consider an irreducible skip-free random walk with transition probability P on $\{0, 1, \ldots, d\}$ started at 0, and suppose d is an absorbing state. Then the hitting time of state d has the generation function

$$\varphi_d(s) = \prod_{i=0}^{d-1}\left[\frac{(1-\lambda_i)s}{1-\lambda_i s}\right],$$

where $\lambda_0, \ldots, \lambda_{d-1}$ are the d non-unit eigenvalues of P.

In particular, if all of the eigenvalues are real and nonnegative, then the hitting time is distributed as the sum of d independent geometric random variables with parameters $1-\lambda_i$.

Theorem 1.2. *For the skip-free birth and death chain:*

Consider an irreducible skip-free birth and death chain with generator Q on $\{0, 1, \ldots, d\}$ started at 0, and suppose d is an absorbing state. Then the hitting time of state d has the Laplace transform

$$\varphi_d(s) = \prod_{i=0}^{d-1}\frac{\lambda_i}{\lambda_i + s},$$

where λ_i are the d non-zero eigenvalues of $-Q$.

In particular, if all of the eigenvalues are real and nonnegative, then the hitting time is distributed as the sum of d independent exponential random variables with parameters λ_i.

2. Proof of Theorem 1.1

Define the transition probability matrix P as

$$P = \begin{pmatrix} r_0 & p_0 & & & & \\ q_{1,0} & r_1 & p_1 & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \\ q_{d-1,0} & q_{d-1,1} & q_{d-1,2} & \cdots & r_{d-1} & p_{d-1} \\ & & & & & 1 \end{pmatrix}_{(d+1)\times(d+1)},$$

and, for $0 \le n \le d-1$, let P_n denote the first $n+1$ rows and first $n+1$ lines of P.

Let $\tau_{i,i+j}$ be the hitting time of state $i+j$ starting from i. By the Markov property, we have

$$\tau_{i,i+j} = \tau_{i,i+1} + \tau_{i+1,i+2} + \cdots + \tau_{i+j-1,i+j}. \tag{2.1}$$

If $f_{i,i+1}(s)$ is the generation function of $\tau_{i,i+1}$, then

$$f_{i,i+1}(s) = \mathbb{E}s^{\tau_{i,i+1}} \quad \text{for } 0 \le i \le d-1.$$

Notice that the random variables on the right hand side of (2.1) are independent, and so we have

$$f_{i,i+j}(s) = f_{i,i+1}(s)\cdot f_{i+1,i+2}(s)\cdots f_{i+j-1,i+j}(s), \quad \text{for } 1 \le j \le d-i.$$

Let

$$g_{0,0}(s) = 1, \qquad g_{i,i+j}(s) = \frac{p_i p_{i+1}\cdots p_{i+j-1}}{f_{i,i+j}(s)} s^j, \quad \text{for } 1 \le j \le d-i.$$

Lemma 2.1. *Define I_n as a $(n+1)\times(n+1)$ identity matrix. We have*

$$g_{0,n+1}(s) = \det(I_n - sP_n), \quad \textit{for } 0 \le n \le d-1. \tag{2.2}$$

Proof. We will give a key recurrence to prove this lemma. By decomposing the first step, the generation function of $\tau_{n,n+1}$ satisfies

$$f_{n,n+1}(s) = r_n s f_{n,n+1}(s) + p_n s + q_{n,n-1} s f_{n-1,n+1}(s) + q_{n,n-2} s f_{n-2,n+1}(s) + \cdots + q_{n,0} s f_{0,n+1}(s). \tag{2.3}$$