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Poisson point processes with detection and rest

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ABSTRACT

Using the Poisson point process we model how the SiPM works and derive the non-linear response formula of the SiPM. Using this non-linear response formula we are able to capture the mean and variance saturation phenomena near the infinity and the linear behavior of the mean and variance near 0.

We also introduce a different model of the SiPM. Under this different model we show the mean and variance saturation phenomena near the infinity and the linear behavior of the mean and variance near 0. We provide some open problems on this different model.

1. Introduction and statement of result

Recently there has been great interest on the development of the Silicon Photomultipliers (SiPM) to use them in the MR compatible PET detectors as well as high energy physics, neutron physics, and bioluminescence. This popularity of the SiPM is mainly due to the characteristics of the SiPM such as its compactness, low operating bias, high gain, fast timing, and non-sensitivity to the magnetic field. See Petasecca et al. (2008) for details.

The SiPM consists of several thousands of cells called Geiger-mode avalanche photodiodes. Once a photon hits a cell, with a certain probability called the PDE, which is denoted by p in this paper for simplicity, the cell produces a large current flow and the external circuitry, known as a quench circuit, senses this current. This large current flow then resets the cell and the cell is ready to detect another incoming photon. In this way the cell produces the on and off states, and the SiPM with several thousands of cells enables us to measure the flux of incoming photons. Even though the SiPM has many benefits, it also has several shortfalls largely due to the limited number of cells in it. So, the SiPM should be carefully designed taking into account the PDE, the number of cells, and many others. See Corsi et al. (2007) for details.

To enhance the performance of the SiPM, a scintillator is attached in front of the SiPM. If a photon hits the scintillator, the scintillator emits a lot of photons and large portion of the emitted photons arrive in the SiPM attached to the scintillator. In order to increase the resolution, the cells of the SiPM should detect as many incoming photons as possible and this can be done by optimizing the PDE and the number of the cells with the considerations of the cell reset time, the number of photons, and the error due to the non-linear response of the SiPM. See Lee et al. (2011) and Stoykov et al. (2007) for details.

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In this paper we model how the SiPM works in a somewhat ideal way and derive the non-linear response formula of the SiPM, Proposition 2. With this non-linear response formula one is able to design the SiPM with high fidelity.

If a photon hits a scintillator at $t = 0$, the scintillator emits a lot of photons. Some of the photons are absorbed to the interior of the scintillator. However a large portion of the photons arrive in the SiPM attached to the scintillator. Let $N_{\text{inc}}(t)$ be the average number of the photons which arrive in the SiPM up to time t and let $f(t) = dN_{\text{inc}}(t)/dt$ be its rate. We assume that the arrival process of the photons in the SiPM is the Poisson point process (PPP) with rate $f(t)$.

(A1) The arrival process of the photons in the SiPM is the PPP with rate $f(t)$.

The SiPM consists of N_{cells} cells. When a photon arrives in the SiPM, we assume that it chooses a cell to hit with equal probability independently.

(A2) The incoming photon independently chooses a cell to hit with equal probability.

Under (A1)–(A2) by the splitting property of the Poisson point process (see Bertsekas and Tsitsiklis, 2008 for details) each arrival process of the photons to a cell is the Poisson point process with rate $g(t) := f(t)/N_{\text{cells}}$ and the arrival process to a cell is independent of another arrival process to a different cell.

If the cell is ready to detect the incoming photon and if the photon hits the cell, we assume that with probability $p :=$ PDE the cell detects it independently.

(A3) The cell detects the incoming photon with probability p independently
if the cell is ready to detect and if the photon hits the cell.

One can think that the cell consists of two regions, the detection region and the dark region. If the photon hits the cell, it lands on the detection region with probability p and the dark region with probability $1 - p$. If the photon lands on the detection region and if the cell is ready to detect, the cell detects this incoming photon. If the photon lands on the detection region but if the cell is not ready to detect, the cell fails to detect the photon. If the photon lands on the dark region, the cell fails to detect. We call a photon which lands on the detection region, a ready-to-detect photon.

Under (A1)–(A3) again by the splitting property of the Poisson point process each arrival process of the ready-to-detect photons to a cell is the Poisson point process with rate $h(t) := pg(t)$ and the arrival process of the ready-to-detect photons to a cell is independent of another arrival process of the ready-to-detect photons to a different cell.

If the cell detects the photon, we assume that it rests an independent exponential time with parameter τ^{-1} (i.e., the mean rest time is τ) before being ready to detect another incoming photon.

(A4) If the cell detects the photon, it rests an independent exponential
time with parameter τ^{-1} to be ready to detect another incoming photon.

Our main result is the explicit calculation of the mean and variance of the number $N(T_0)$ of photons detected up to time T_0 by a particle cell.

Theorem 1. *Assume that* (A1)–(A4) *hold.*

(1) *Let $N(T_0)$ be the number of photons detected up to time T_0 by a particle cell. Then, the mean and variance of $N(T_0)$ are given by*

$$E(N(T_0)) = \int_0^{T_0} H_1(t)dt + \int_0^{T_0}\int_0^t H_2(t,s)dsdt, \tag{1.1}$$

$$\text{Var}\,(N(T_0)) = (E(N(T_0)))(1 - (E(N(T_0)))) + 2\int_0^{T_0} H_1(t)H_3(t)dt + 2\int_0^{T_0}\int_0^t H_2(t,s)H_3(t)dsdt, \tag{1.2}$$

where $H_1(t), H_2(t,s), H_3(t)$ are given by

$$H_1(t) := h(t)e^{-\int_0^t h(u)du}, \tag{1.3}$$

$$H_2(t,s) := h(t)h(s)e^{-\int_s^t h(u)du}\left(1 - e^{-\frac{t-s}{\tau}}\right), \tag{1.4}$$

$$H_3(t) := \int_t^{T_0}\int_t^{t'} H_2(t',s')ds'dt' + \int_t^{T_0} h(t')e^{-\int_t^{t'} h(u)du}\left(1 - e^{-\frac{t'-t}{\tau}}\right)dt'. \tag{1.5}$$

(2) *Let $N_{\text{det}}(T_0)$ be the number of photons detected up to time T_0 by the SiPM. Then, the mean and variance of $N_{\text{det}}(T_0)$ are given by*

$$EN_{\text{det}}(T_0) = N_{\text{cells}}E(N(T_0)), \tag{1.6}$$

$$\text{Var}\,(N_{\text{det}}(T_0)) = N_{\text{cells}}\text{Var}\,(N(T_0))\,. \tag{1.7}$$

In Section 2 we prove Theorem 1. In Section 3 we discuss some consequences of Theorem 1. In particular, we derive the non-linear response formula of the SiPM, Proposition 2. In Section 4 we present a different model of the SiPM with some open problems.