Statistics and Probability Letters 80 (2010) 42–49

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Sensitivity analysis for averaged asset price dynamics with gamma processes

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ARTICLE INFO

Article history:
Received 1 April 2009
Received in revised form 15 September 2009
Accepted 23 September 2009
Available online 30 September 2009

MSC:
60E07
60G51
91B70
60H07

ABSTRACT

The main purpose of this paper is to derive unbiased Monte Carlo estimators of various sensitivity indices for an averaged asset price dynamics governed by the gamma Lévy process. The key idea is to apply a scaling property of the gamma process with respect to the Esscher density transform parameter. Our framework covers not only the continuous Asian option, but also European, discrete Asian, average strike Asian, weighted average, spread options, and geometric average Asian options. Numerical results are provided to illustrate the effectiveness of our formulas in Monte Carlo simulations, relative to finite difference approximation.

1. Introduction

In mathematical finance, it has been widely known that the logarithmic derivatives of the density function of stochastic differential equations correspond essentially to the so-called Greeks, that is, sensitivity indices with respect to various model parameters of asset price dynamics. The pioneer work in this direction is of Fournié et al. (1999), whose approach is based upon the integration-by-parts formula developed in the Malliavin calculus on the Wiener space. They provide a systematic approach to the derivation of various Greeks formulas for a relatively general diffusion asset price dynamics. Since then, there has long been a natural and nontrivial question as to whether a similar Malliavin calculus approach is applicable in the case of jump processes.

The investigation of the Malliavin calculus for jump processes is initiated in Bismut (1983) based upon the Girsanov density transform with a view towards the study of the existence and the smoothness of the density. Various types of Malliavin calculus and logarithmic derivatives have been studied on the Poisson space or on the Wiener–Poisson space. In particular, the existence of weights for the logarithmic derivatives dates back to the work of Bichteler et al. (1987), while the uniqueness and the closed forms of the weight have not been established in the general framework.

For practical considerations, however, it is most important to have closed, but not necessarily unique, weights on hand, in order to design an efficient Monte Carlo evaluation. In this direction, Greeks formulas are obtained in Davis and Johansson (2006) and Cass and Friz (2007) for jump diffusion processes. Their approach consists of conditioning on the jump component and then performing the Malliavin calculus techniques on the diffusion component. In other words, their models are required to be a superposition of independent jump and diffusion components, where the diffusion one must not be degenerate. On the other hand, El-Khatib and Privault (2004) applied the Malliavin calculus focused on the Poisson arrival times due to Carlen and Pardoux (1990), while Bally et al. (2007) took a unified approach considering the derivatives with respect to

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0167-7152/$ – see front matter
doi:10.1016/j.spl.2009.09.010

both the Poisson arrival times and the amplitude of the jumps. Kawai and Kohatsu-Higa (in press) derives formulas for Lévy process models of time-changed Brownian motion type, by employing the Gaussian Malliavin calculus conditionally on the time-changing process. Takeuchi (under revision) studied the same problem for the stochastic differential equations with jumps via a martingale approach. Kawai and Takeuchi (under revision) studied the computation of the Greeks of European payoffs for an asset price dynamics defined with gamma processes and with Brownian motions, possibly time-changed by one of the gamma processes. In Kawai and Takeuchi (under revision), various well known financial models, such as the Black–Scholes model and the variance gamma model of Madan et al. (1998), are within its framework. The key idea there is to apply a scaling property of the gamma *distribution* with respect to the Esscher transform.

In this paper, we take an approach again based on the scaling property, not of the gamma distribution as in Kawai and Takeuchi (under revision), but of the gamma Lévy *process*. This, rather simple, extension widens the applicability to the *averaged* asset price dynamics defined with the gamma process. A typical example of interest is the (continuous) Asian option, which looks at the time average of underlying asset price dynamics, unlike only the terminal value in the European options. Meanwhile, many others, such as European, discrete Asian, average strike Asian, weighted average and spread options, can easily be accommodated in our framework. In addition, Asian options of geometric average type are also within our scope. This work can also be thought of as a considerable generalization of Kawai and Takeuchi (under revision), in the sense that some of the European Greeks formulas there can be recovered from our results. Finally, our approach taken in this paper can also be applied in various stochastic systems involving gamma processes.

The rest of the paper is organized as follows. Section 2 recalls generalities on the gamma process and introduces our asset price dynamics model with assumptions to be imposed on its characterizing parameters. In Section 3, we derive formulas of the Greeks for discontinuous payoff function of averaged asset price dynamics defined with the gamma process. The derivation of our formulas entails rather lengthy proofs of somewhat routine nature. To avoid overloading the paper, we omit nonessential details in some instances. We close this study with some numerical results to illustrate remarkable improvements in Monte Carlo simulations in terms of estimator variance relative to the finite difference estimation.

2. Preliminaries

Let us begin with general notations which will be used throughout the text. For each positive integer k, ∂_k indicates the partial derivative with respect to k-th argument. We fix $(\Omega, \mathcal{F}, \mathbb{P})$ as our underlying the probability space. We denote by $\mathbb{E}_{\mathbb{P}}$ the expectation under the probability measure $\mathbb{P}$. We denote by $\delta_x(dy)$ the Dirac delta measure on $\mathbb{R}$ with concentration at $x \in \mathbb{R}$, while $B_\epsilon := (-\epsilon, +\epsilon)$ is the open ball around the origin with radius $\epsilon > 0$. Let $\{Y_t : t \geq 0\}$ be a one-sided pure-jump Lévy process in $[0, +\infty)$ with the Lévy measure

$$\nu(\mathrm{d}z) = a\frac{\mathrm{e}^{-bz}}{z}\,\mathrm{d}z, \quad z \in (0, +\infty),$$

where $a > 0$ and $b > 0$. The stochastic process $\{Y_t : t \geq 0\}$ is then called a *gamma process* with parameter (a, b), whose marginal has the gamma distribution with characteristic function

$$\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\mathrm{i}yY_t}\right] = \exp\left[t\int_{(0,+\infty)}\left(\mathrm{e}^{\mathrm{i}yz} - 1\right)\nu(\mathrm{d}z)\right] = \left(1 - \frac{\mathrm{i}y}{b}\right)^{-at}.$$

Moreover, the marginal density function at time t is given in the form

$$f_t^{\mathbb{P}}(y) = \frac{b^{at}}{\Gamma(at)}\,y^{at-1}\mathrm{e}^{-by}, \quad y \in (0, +\infty), \tag{1}$$

where $\Gamma(p)$ is the gamma function of order $p > 0$.

Throughout the paper, we fix $T > 0$. Define $\Lambda := \left\{\lambda \in \mathbb{R} : \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{\lambda Y_1}] < +\infty\right\} = (-\infty, b)$. For $\lambda \in \Lambda$, define a new probability measure $\mathbb{Q}_\lambda$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}_\lambda}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} := \frac{\mathrm{e}^{\lambda Y_T}}{\mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{\lambda Y_T}\right]} = \exp\left[\lambda Y_T - \varphi_T(\lambda)\right],$$

where $\varphi_T(\lambda) := aT\ln\left(b/(b-\lambda)\right)$ and where $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration generated by $\{Y_t : t \in [0, T]\}$. This measure change is the simplest form of the Girsanov transformation, often called the Esscher transform in mathematical finance and actuarial science. The following is the key tool for our discussions.

Lemma 2.1. *The laws of the process Y under $\mathbb{Q}_\lambda$ and of the process $Y^{(\lambda)} := bY/(b-\lambda)$ under $\mathbb{P}$ are identical.*

Proof. It is well known that for Lévy processes, the identity of finite dimensional distributions implies the identity of random processes in law. Fix $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$ and $\zeta_1, \ldots, \zeta_n \in \mathbb{R}$ arbitrarily. Due to the independence of the increments