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A representation theorem for generators of BSDEs with finite or infinite time intervals and linear-growth generators[☆]

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ABSTRACT

Under the most elementary conditions on stochastic differential equations and some milder conditions on backward stochastic differential equations with finite or infinite time intervals and linear-growth generators, a representation theorem of generators and a converse comparison theorem of solutions are established in this paper.

1. Introduction

It is well known from Pardoux and Peng (1990) that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE for short in the remainder of the paper) with a finite time interval of type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad 0 \le t \le T < +\infty, \tag{1.1}$$

provided that the generator g is Lipschitz in both variables y and z, and that the terminal data ξ and the process $(g(t,0,0))_{t\in[0,T]}$ are square integrable. The triple (ξ, T, g) is called the parameters of BSDE (1.1). We denote the unique solution mentioned above by $(y_t(\xi,T,g), z_t(\xi,T,g))_{t\in[0,T]}$, and often denote $y_t(\xi,T,g)$ by $\mathcal{E}^g_{t,T}[\xi]$ for each $t \in [0,T]$.

One of the achievements of BSDE theory is the comparison theorem (see El Karoui et al., 1997). Recently, some papers have been devoted to studying the converse comparison theorem. For studying the converse comparison theorem, Briand et al. (2000) established the following representation theorem for generators of BSDEs with a finite time interval: for each $(t,y,z) \in [0,T[\times \mathbf{R}^{1+d}$,

$$\lim_{n\to\infty} n\{\mathcal{E}^g_{t,t+1/n}[y + z\cdot(B_{t+1/n} - B_t)] - y\} = g(t,y,z) \tag{1.2}$$

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holds true in L^2 for g satisfying two additional assumptions that $(g(t, y, z))_{t\in[0,T]}$ is continuous with respect to t for each (y, z) and $\mathbf{E}\left[\sup_{0\le t\le T} |g(t, 0, 0)|^2\right] < +\infty$.

From then on, the interest in weakening and eliminating the above two additional assumptions has increased. Let us especially mention the following contributions: without the two additional assumptions for generators of BSDEs, Jiang (2008) successfully proved that (1.2) holds true for almost every $t \in [0, T[$ in L^p $(1 \le p < 2)$. Fan and Hu (2008) proved that (1.2) holds also true in the space of processes. Furthermore, Lepeltier and San Martín (1997) proved the existence of minimal solutions for BSDEs with continuous and linear-growth generators, and Fan and Jiang (2010) established a representation theorem for generators of this kind of BSDEs in the space of processes. It is worth mentioning that all these papers dealt with BSDEs with finite time intervals.

On the other hand, under a non-uniform Lipschitz condition (in t) of the generator g with respect to (y, z), Chen and Wang (2000) proved that BSDE (1.1) admits still a unique solution when $T = +\infty$. Based on this result, Fan et al. (2011) proved the existence of minimal solutions for BSDEs with finite or infinite time intervals and continuous and linear-growth generators. Then, a natural question is asked:

Can we establish a representation theorem for generators of BSDEs with finite or infinite time intervals and linear-growth generators?

This paper solves this problem. The remainder of this paper is organized as follows. In Section 2, after introducing some preliminaries, we state our main results— Theorems 2.1–2.3. In Section 3, we first establish some lemmas and propositions and then give the proof of our main results.

2. Preliminaries and the statement of main results

In this section, we will first introduce some definitions, assumptions, notations and then state the main results of this paper.

Suppose that $0 < T \le +\infty$ and let $[0, T]$ mean $[0, +\infty[$ when $T = +\infty$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(B_t)_{t\ge 0}$ be a standard d-dimensional Brownian motion on this space such that $B_0 = 0$. Let $(\mathcal{F}_t)_{t\ge 0}$ be the natural σ-algebra filtration generated by $(B_t)_{t\ge 0}$. We always assume that $(\mathcal{F}_t)_{t\ge 0}$ is right continuous and complete. For each $p \in [1, 2]$, we define the following usual space of random variables:

$$L^p(\Omega, \mathcal{F}_T, P) = \{X \in \mathbf{R} : X \text{ all } \mathcal{F}_T\text{-measurable; } \mathbf{E}[|X|^p] < +\infty\}.$$

In this paper, we always work in the space $(\Omega, \mathcal{F}_T, P)$, where T may be finite or infinite. For any positive integer d, let $|z|$ denote the Euclidean norm of $z \in \mathbf{R}^d$. $\mathbf{R}^{n\times d}$ is identified with the space of real matrices with n rows and d columns, we define $|z|^2 = \mathrm{trace}(zz^*)$ if $z \in \mathbf{R}^{n\times d}$. We also define the following usual spaces of processes:

$$\mathcal{S}^2_{\mathcal{F}}(0, T; \mathbf{R}) = \left\{(\psi_t)_{t\in[0,T]} \in \mathbf{R} : \psi_t \text{ all continuous and } (\mathcal{F}_t)\text{-progressively measurable; } \mathbf{E}\left[\sup_{0\le t\le T} |\psi_t|^2\right] < +\infty\right\};$$

$$\mathcal{H}^2_{\mathcal{F}}(0, T; \mathbf{R}^d) = \left\{(\psi_t)_{t\in[0,T]} \in \mathbf{R}^d : \psi_t \text{ all } (\mathcal{F}_t)\text{-progressively measurable; } \mathbf{E}\left[\int_0^T |\psi_t|^2 \mathrm{d}t\right] < +\infty\right\}.$$

Let $b(\cdot, \cdot, \cdot) : \Omega \times [0, T] \times \mathbf{R}^n \to \mathbf{R}^n$, $\sigma(\cdot, \cdot, \cdot) : \Omega \times [0, T] \times \mathbf{R}^n \to \mathbf{R}^{n\times d}$ be two functions such that $b(\cdot, \cdot, x)$ and $\sigma(\cdot, \cdot, x)$ are both $(\mathcal{F}_t)$-progressively measurable for any $x \in \mathbf{R}^n$. Let b and σ also satisfy the following assumptions (H1) and (H2).

(H1) There exists a constant $K_1 \ge 0$ such that $\mathrm{d}P \times \mathrm{d}t$-a.e.,

$$\forall x, y \in \mathbf{R}^n, \quad |b(\omega, t, x) - b(\omega, t, y)| + |\sigma(\omega, t, x) - \sigma(\omega, t, y)| \le K_1|x - y|.$$

(H2) There exists a constant $K_2 \ge 0$ such that $\mathrm{d}P \times \mathrm{d}t$-a.e.,

$$\forall x \in \mathbf{R}^n, \quad |b(\omega, t, x)| + |\sigma(\omega, t, x)| \le K_2(1 + |x|).$$

Given $(t, x) \in [0, T[\times\mathbf{R}^n$, by classical stochastic differential equation (SDE in short) theory, we know that for each $\bar{T} \in [t, T[$, the following SDE

$$X_s = x + \int_t^s b(u, X_u)\mathrm{d}u + \int_t^s \sigma(u, X_u)\mathrm{d}B_u, \quad s \in [t, \bar{T}];\ X_s = x,\ s \in [0, t] \tag{2.1}$$

has a unique s-continuous solution, denoted by $(X_s^{t,x})_{s\in[0,\bar{T}]}$, with the properties that $(X_s^{t,x})_{s\in[0,\bar{T}]}$ is $(\mathcal{F}_s)$-adapted and

$$\mathbf{E}\left[\sup_{0\le s\le \bar{T}} |X_s^{t,x}|^2\right] < +\infty, \quad \text{and} \quad s \to \mathbf{E}|X_s^{t,x} - x|^2, \quad s \in [0, \bar{T}], \text{ is continuous.} \tag{2.2}$$

The generator g of BSDE (1.1) is a function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ such that the process $(g(t, y, z))_{t\in[0,T]}$ is $(\mathcal{F}_t)$-progressively measurable for each $(y, z) \in \mathbf{R} \times \mathbf{R}^d$ and g satisfies the following assumptions (A1) and (A2), where $0 < T \le +\infty$.