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Fixed jumps of additive processes

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ABSTRACT

A process in a Euclidean space is called an additive process if it has independent increments. We recall the classical Lévy–Itô representation for additive processes without fixed jumps, and describe how fixed jumps were handled in the classical literature. Our main result is an extended Lévy–Itô formula in which the fixed jumps are expressed in a canonical and convenient form.

1. Introduction

Let x_t, $t \in \mathbb{R}_+ = [0, \infty)$, be a process in $\mathbb{R}^d$ with rcll paths (right continuous paths with left limits). It is called an additive process if it has independent increments in the sense that for $s < t$, $x_t - x_s$ is independent of $\mathcal{F}_s^x$, where $\{\mathcal{F}_t^x\}$ is the natural filtration of x_t. It is said to have a fixed jump at time $t > 0$ if $P(x_t \neq x_{t-}) > 0$. A rcll process without fixed jump is called stochastically continuous. The classical Lévy–Itô representation expresses a stochastically continuous additive process as a sum of a non-random drift, a time inhomogeneous Brownian motion and independent jumps counted by a Poisson random measure. This formula will be recalled more precisely in Section 2.

In general, the fixed jumps of an additive process do not form a convergent part of the process. Lévy (1965) proved that after subtracting suitable centralizing constants, the sum of fixed jumps converge, but the constants were not identified. More explicit approaches were taken by Itô (1969) and Loève (1978), but their centralizing constants are non-canonical and quite complicated, see Section 2 for more details.

The main purpose of this paper is to show that the centralizing constants of fixed jumps may be taken to be their truncated means. The result is given in the form of an extended Lévy–Itô representation for additive processes, in which the fixed jumps are expressed in a canonical and convenient fashion, see Section 3. This form of the formula appears to be new. A direct proof, or a proof based on the centralizing methods in the literature cited above, does not seem to be easy. However, a key step of the proof is already obtained in Jacod and Shiryaev (2003) for establishing a Fourier transform of additive processes. This allows us to provide a rather quick proof.

2. The classical Lévy–Itô representation

We first mention some standard notation. For $x \in \mathbb{R}^d$, let $|x|$ be its Euclidean norm. For two real numbers a and b, let $a \wedge b = \min(a, b)$. For any function f and measure μ, the integral $\int f d\mu$ may be written as $\mu(f)$. The Borel σ-field of a topological space X is denoted $\mathcal{B}(X)$.

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A continuous path b_t in $\mathbb{R}^d$ with $b_0 = 0$ (origin) will be called a drift. Let $B_t = (B_t^1, \ldots, B_t^d)$ be a continuous d-dim Gaussian process of mean 0 and with independent increments. Its distribution is determined completely by its covariance matrix function $A_{ij}(t) = E(B_t^i B_t^j)$. When $A_{jk}(t) = at\delta_{ij}$ for some constant $a > 0$, B_t is a Brownian motion in $\mathbb{R}^d$. In general, B_t will be called an inhomogeneous Brownian motion in $\mathbb{R}^d$.

A random measure N on $\mathbb{R}_+ \times \mathbb{R}^d$, taking values from nonnegative integers, including ∞, will be called a counting measure. We will assume all counting measures N also satisfy that $E[N([0, t] \times \cdot)]$ is a σ-finite measure on $\mathbb{R}^d$ for any $t > 0$, and almost surely,

$$N(\{t\} \times \mathbb{R}^d) \le 1 \quad \text{for all } t > 0, \quad \text{and} \quad N(\{0\} \times \mathbb{R}^d) = N(\mathbb{R}_+ \times \{0\}) = 0. \tag{1}$$

A counting measure counts points in $\mathbb{R}^d$ at any time t, and (1) means that at each t, there is at most one point, and no point at time 0 and no point at origin 0. The measure $\eta = E[N(\cdot)]$ on $\mathbb{R}_+ \times \mathbb{R}^d$ is called the intensity measure of N. The measure-valued function $\eta_t = \eta([0, t] \times \cdot)$ is nondecreasing and right continuous in the sense that $\eta_s \le \eta_t$ for $s < t$ and $\eta_t \downarrow \eta_s$ as $t \downarrow s$, and its left limit η_{t-} at any $t > 0$ is defined as the nondecreasing limit of η_s as $s \uparrow t$.

A counting measure N with intensity η is called a Poisson measure if η is continuous in time, that is, $\eta_{t-} = \eta_t$ (or equivalently $\eta(\{t\} \times \mathbb{R}^d) = 0$) for $t > 0$, and it has independent increments in the sense that the measure-valued process $N_t = N([0, t] \times \cdot)$ does so. By Theorem 12.10 in Kallenberg (2002), for any disjoint $B_1, B_2, \ldots, B_k \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$, $N(B_1), N(B_2), \ldots, N(B_k)$ are independent Poisson random variables of means $\eta(B_1), \eta(B_2), \ldots, \eta(B_k)$. Here, a Poisson random variable of mean 0 or ∞ is defined to be 0 or ∞. Note that our definition of Poisson measures follows Jacod and Shiryaev (2003) and is not as general as given in Kallenberg (2002).

We now summarize some simple facts about the integration of a Poisson measure N, which can be easily proved. Let η be the intensity of N. The compensated form of N is $\tilde{N} = N - \eta$. For $p > 0$, let F^p be the space of Borel functions f on $\mathbb{R}^d$ such that $\eta_t(|f|^p) < \infty$ for any $t > 0$. For $f \in F^1 \cap F^2$, the integral $\tilde{N}_t(f) = N_t(f) - \eta_t(f)$ is well defined and is an L^2-martingale (under the natural filtration of N_t) with $E[\tilde{N}_t(f)^2] = \eta_t(f^2)$. For $f \in F^2$, $N_t(f) - \eta_t(f)$ may not be well defined, but $\tilde{N}_t(f)$ is defined as the L^2-limit of $\tilde{N}_t(f_n)$ for any sequence $f_n \in F^1 \cap F^2$ with $\eta_t((f - f_n)^2) \to 0$. Then by Doob's norm inequality for martingales, almost surely, $\tilde{N}_t(f_n) \to \tilde{N}_t(f)$ uniformly in t in any bounded interval.

We now present the modern form of Lévy–Itô representation, see Theorem 15.4 in Kallenberg (2002) or Theorem 19.2 in Sato (1999). If x_t is an additive process in $\mathbb{R}^d$ with $x_0 = 0$ and without fixed jumps, then there is a triple (b, B, N), unique almost surely, of a drift b_t in $\mathbb{R}^d$, an inhomogeneous Brownian motion B_t in $\mathbb{R}^d$, and an independent Poisson measure N on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity η satisfying

$$\int_{|x|>1} \eta_t(dx) < \infty \quad \text{and} \quad \int_{|x|\le 1} |x|^2 \eta_t(dx) < \infty \tag{2}$$

for any $t > 0$, such that

$$x_t = b_t + B_t + \int_{|x|\le 1} x\tilde{N}_t(dx) + \int_{|x|>1} xN_t(dx). \tag{3}$$

Conversely, for any triple (b, B, N) with properties stated above, x_t given by (3) is an additive process in $\mathbb{R}^d$ with $x_0 = 0$ and without fixed jumps.

Note that the first integral in (3) is the limit of $\int_{\varepsilon<|x|\le 1} xN_t(dx) - \int_{\varepsilon<|x|\le 1} x\eta_t(dx)$ as $\varepsilon \to 0$ and the convergence is uniform for bounded t almost surely. Because $N_t(\{x \in \mathbb{R}^d;\ |x| > \varepsilon\})$ is a finite Poisson random variable, both $\int_{\varepsilon<|x|\le 1} xN_t(dx)$ and the second integral in (3) are sums of finitely many nonzero terms almost surely. Because η is continuous in time, $\int_{\varepsilon<|x|\le 1} x\eta_t(dx)$ is continuous in t, it is then easy to see from (3) that the Poisson random measure N is the jump counting measure of process x_t defined by

$$N([0, t] \times B) = \#\{u \in (0, t];\ \Delta x_u \in B \text{ and } \Delta x_u \ne 0\}, \quad (\Delta x_u = x_u - x_{u-}), \tag{4}$$

the number of the jumps in B during the time interval $[0, t]$, for $t > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

Lévy (1965) proved that the fixed jumps, after subtracting suitable constants, form a convergent part of process x_t. The approach in Itô (1969) is more explicit and is now described here. The central value $\gamma(x)$ of a random variable x is defined (by Doob) as the unique real number γ such that $E[\tan^{-1}(x - \gamma)] = 0$, and its dispersion is defined as $\delta(x) = -\log E[e^{-|x-y|}]$, where y is an independent copy of x. Although they do not share the usual properties of the mean and variance, it holds that $\gamma(x + r) = \gamma(x) + r$ for any real number r and $\delta(x + y) \ge \delta(x)$ for any random variable y independent of x. Moreover, if x_n are independent with $s_n = x_1 + \cdots + x_n$, then $s_n - \gamma(s_n)$ converges almost surely if and only if $\delta(s_n)$ is bounded. For an additive process x_t, its fixed jumps are countably many and may be ordered as a sequence. Let s_n be the sum of the first n fixed jumps Δx_s for $s \le t$. Because $\delta(s_n) \le \delta(x_t)$, the centralized sum $s_n - \gamma(s_n)$ converges to a process w_t as $n \to \infty$. Then $x_t - w_t - \gamma(x_t)$ has no fixed jumps to which the Lévy–Itô formula may be applied.