Statistics and Probability Letters 83 (2013) 431–435

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Absolutely continuous random power series in reciprocals of Pisot numbers

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ARTICLE INFO

Article history:
Received 12 April 2012
Received in revised form 8 October 2012
Accepted 8 October 2012
Available online 5 November 2012

MSC:
26A46
26A30
28A80

Keywords:
Random power series
Absolute continuity
Pisot numbers

ABSTRACT

We construct a continuum of shift invariant Borel probability measures on $\{-1, 1\}^{\mathbb{N}}$ such that corresponding random power series in reciprocals of Pisot numbers are absolutely continuous with respect to the Lebesgue measure.

1. Introduction

Let $\beta \in (0.5, 1)$ be the reciprocal of a Pisot number $\alpha \in (1, 2)$; this is an algebraic integer with all conjugates inside the unit circle; see Bertin et al. (1992). Let $\mathfrak{M}$ be the compact and convex space of all shift invariant Borel probability measures on $\{-1, 1\}^{\mathbb{N}}$ with the weak* topology; see Walters (1982). $\mu \in \mathfrak{M}$ induces a random power series via

$$X_\beta = \sum_{i=1}^{\infty} X_i \beta^i,$$

where (X_i) is the stationary process taking values in $\{-1, 1\}$ according to μ. Let μ_β be the Borel probability measure on the real line describing the distribution of X_β: $\mu_\beta(B) = P(X_\beta \in B)$. It is a classical result that goes back to Paul Erdös that μ_β is singular if (X_i) is a Bernoulli process. In fact Erdös (1939) proved the result for the equal weight Bernoulli measure. The proof can easily be generalized to arbitrary homogeneous (see Lalley (1999)) and even inhomogeneous (see Bisbas and Neunhäuserer (2011)) Bernoulli processes. We are interested here in measures $\mu \in \mathfrak{M}$ such that μ_β is absolutely continuous with respect to the Lebesgue measure. We consider the set

$$\mathfrak{A}_\beta = \{\mu \in \mathfrak{M} \mid \mu_\beta \text{ is absolutely continuous}\}$$

and prove here the following:

Theorem 1.1. *$\mathfrak{A}_\beta$ is a continuum: a compact and convex space with more than one element.*

It is easy to infer from known results that $\mathfrak{A}_\beta$ contains the Parry measure μ with measure theoretic entropy $H(\mu) = \log(\beta^{-1})$. We give detailed references in the proof of Proposition 2.1. Furthermore using the Garsia entropy we obtain that

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doi:10.1016/j.spl.2012.10.024

the space $\mathfrak{A}_\beta$ is compact and convex; see Corollary 2.1. The main aim of this work is to construct a measure $\mu \in \mathfrak{A}_\beta$ with measure theoretic entropy $H(\mu) > \log(\beta^{-1})$. This proves the theorem; see Section 3.

2. Garsia entropy

We define a coding map $\pi : \{-1, 1\}^{\mathbb{N}} \longmapsto \mathbb{R}$ as follows:

$$\pi((s_i)) = \sum_{i=1}^{\infty} s_i \beta^i.$$

$\mu \in \mathfrak{M}$ induces a Borel probability measure on $\mathbb{R}$ through $\mu_\beta = \pi(\mu) = \mu \circ \pi^{-1}$, which obviously is the distribution of the random power series X_β given in the introduction.

Let $\mathcal{C}_n$ be the partition of $\{-1, 1\}^{\mathbb{N}}$ into cylinder sets of length $n \geq 1$. The Shannon entropy of a partition with respect to μ is given by

$$H_\mu(\mathcal{C}_n) = -\sum_{C \in \mathcal{C}_n} \mu(C) \log(\mu(C)).$$

As a convention, we use logarithms to base 2 throughout the work. Since $\mathcal{C}_{n+m} = \mathcal{C}_n \vee \sigma^{-n}\mathcal{C}_m$ for the shift map σ and the join of partitions $\vee$, it is easy to check that $H_\mu(\mathcal{C}_n)$ is a subadditive sequence for shift invariant measures $\mu \in \mathfrak{M}$. Hence the measure theoretic entropy of μ

$$H(\mu) = \lim_{n \longmapsto \infty} \frac{H_\mu(\mathcal{C}_n)}{n}$$

exists. Now we define a sequence of relations $\sim_n$ on $\{-1, 1\}^{\mathbb{N}}$ by

$$(s_i) \sim_n (t_i) :\Leftrightarrow \sum_{i=1}^{n} s_i \beta^i = \sum_{i=1}^{n} t_i \beta^i.$$

Obviously these are equivalence relations. Let $\mathcal{P}_n$ be the partition of $\{-1, 1\}^{\mathbb{N}}$ induced by $\sim_n$. Observe that the partition $\mathcal{P}_n \vee \sigma^{-n}\mathcal{P}_m$ is finer than $\mathcal{P}_{n+m}$. Hence by well known properties of the partition entropy, $H_\mu(\mathcal{P}_n)$ is a subadditive sequence; see Walters (1982). Hence the Garsia entropy

$$G_\beta(\mu) = \lim_{n \longmapsto \infty} \frac{H_\mu(\mathcal{P}_n)}{n}$$

exists; compare with the papers of Garsia (1962, 1963). Note that the partition $\mathcal{C}_n$ is finer than the partition $\mathcal{P}_n$. Hence $G(\mu) \leq H(\mu)$. To be more precise we have

$$H(\mu) = G_\beta(\mu) + \lim_{n \longmapsto \infty} \frac{H_\mu(\mathcal{C}_n \mid \mathcal{P}_n)}{n},$$

where $H_\mu(\mathcal{C}_n \mid \mathcal{P}_n)$ is the conditional entropy; see Walters (1982) for the definition. The first result on the Garsia entropy that we will use is:

Proposition 2.1. *The map $\mu \longmapsto G_\beta(\mu)$ is upper semicontinuous and affine with maximum $\log(\beta^{-1})$ on $\mathfrak{M}$.*

Proof. Upper semicontinuity and being affine follow from Proposition 6.4 of Neunhäuserer (2007), where we proved the result in a more general setting. Furthermore we know that the number of elements in the partition $\mathcal{P}_n$ is bounded by $\sharp\mathcal{P}_n \leq C\beta^{-n}$ for a constant $C > 0$ depending only on β; see Garsia (1962) or Lalley (1999). Hence

$$H_\mu(\mathcal{P}_n) \leq \log(\sharp\mathcal{P}_n) \leq \log(C) + n\log(\beta^{-1}),$$

which implies $G_\beta(\mu) \leq \log(\beta^{-1})$. Now consider the β-subshift $X_\beta \subseteq \{-1, 1\}^{\mathbb{N}}$ with topological entropy $\log(\beta^{-1})$; see Definition 3.1 and Corollary 3.6 of Ito and Takahashi (1974). If $\beta^{-1} \in (1, 2)$ is a Pisot number this subshift is known to be Markovian by Theorem 2 of Ito and Takahashi (1974). Let μ be the Parry measure on X_β—this is the measure of full entropy $H(\mu) = \log(\beta^{-1})$; see Parry (1964). It is known that $\pi : X_\beta \longmapsto \mathbb{R}$ is invertible up to a countable set of sequences; see Proposition 3.2 of Ito and Takahashi (1974). Hence there are no relations $\sim_n$ on X_β and we have $G_\beta(\mu) = H(\mu) = \log(\beta^{-1})$ for the Parry measure μ. □

Now we state a characterization of absolutely continuous measures μ_β using the Garsia entropy, which is implicitly contained in the work of Lalley (1999).

Proposition 2.2. *For $\mu \in \mathfrak{M}$ the measure μ_β is absolutely continuous if and only if $G_\beta(\mu) = \log(\beta^{-1})$.*

Proof. From Proposition 3 of Lalley (1999) we know that

$$\dim_H \mu_\beta \leq -G_\beta(\mu)/\log\beta$$

for the Hausdorff dimension $\dim_H$ of the measure μ_β. If μ_β is absolutely continuous, we have $\dim_H \mu_\beta = 1$ and hence $G_\beta(\mu) = \log\beta^{-1}$. On the other hand using the proof of Proposition 5 of Lalley (1999) which essentially goes back to