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An almost sure local limit theorem for Markov chains

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ABSTRACT

An almost sure local limit theorem for Markov chains is investigated.

1. Introduction and results

Let $\{\xi_k\}_{k\in\mathbb{Z}}$, $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots, k, \dots\}$, be a strictly stationary Markov chain defined on some probability space $(\Omega, \mathcal{F}, P)$. For Borel functions f, $X_k = f(\xi_k)$ defines a family of strictly stationary sequences. Let $S_n = \sum_{k=1}^n X_k$, $n \in \mathbb{N} = \{1, 2, \dots, n, \dots\}$; $\mathbb{R}$ is the real line. Denote by T the usual shift operator on $\mathbb{R}^{\mathbb{Z}}$, i.e., for $\omega := (\omega_k; k \in \mathbb{Z}) \in \mathbb{R}^{\mathbb{Z}}$, the element $T\omega \in \mathbb{R}^{\mathbb{Z}}$ is given by $(T\omega)_k = \omega_{k+1}$, $k \in \mathbb{Z}$. We call $\{\xi_k\}$ mixing (ergodic) if T is mixing (ergodic); see Ash, 2000, Chapter 8 or Bradley, 2007, Chapter 2. In this work, if not stated otherwise, we assume that the state space $\mathbb{S}$ of $\{\xi_k\}$ is countable, so by Theorem 7.7 on p. 212 in Vol. I of Bradley (2007) it follows that $\{\xi_k\}$ is mixing iff it is irreducible and aperiodic (or equivalently β-mixing). In the case of a strictly stationary Markov chain whose state space is a finite set, $\{\xi_k\}$ is mixing iff it is at least exponentially fast ψ-mixing (cf. Bradley (2007, Vol. I, Theorem 7.14, p. 220)).

Suppose that the values of S_n are all of the form $na + kd$, $k \in \mathbb{Z}$, with d being the maximal span. We say that $\{\xi_k\}$ satisfies the normal local limit theorem (LLT) if there exist sequences $\{a_n\}$, $\{b_n\}$, $b_n \to \infty$, such that

$$b_n P(S_n = \kappa_n) \to_n \mathfrak{n}(\kappa) := \frac{1}{\sqrt{2\pi}} e^{\frac{-\kappa^2}{2}} \quad \text{as } n \to \infty,$$

where the sequence κ_n, $n \in \mathbb{N}$, of the form $na + kd$, satisfies

$$\lim_{n\to\infty} \frac{\kappa_n - a_n}{b_n} = \kappa.$$

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The LLT for finite state Markov chains was investigated in Pepper (1927), Bityuckov (1948), Kolmogorov (1949), Manevič (1953) (see also Gnedenko, 1988, Chapter 3, Section 20, pp. 116–122). For countable state Markov chains satisfying $E(\xi_1^2) < \infty$, the (normal) LLT is discussed in Nagaev (1957, 1961, 1963), and Séva (1995) while the case $E(\xi_1^2) = \infty$ is analyzed in Aaronson and Denker (2001) and Szewczak (2008b).

We say that $\{\xi_k\}$ satisfies the almost sure (normal) local limit theorem (ASLLT) if there exist sequences $\{a_n\}$, $\{b_n\}$, $b_n \to \infty$, such that

$$\frac{1}{\ln n}\sum_{\nu=1}^{n}\frac{b_\nu}{\nu}I_{[S_\nu=\kappa_\nu]} \overset{\text{a.s.}}{\to} \mathfrak{n}(\kappa) \quad \text{as } \frac{\kappa_\nu - a_\nu}{b_\nu} \to_\nu \kappa,$$

where the κ_ν are of the form $\nu a + kd$. For the case of independent, identically distributed random variables, the normal ASLLT is studied in Chung and Erdős (1951), Csáki et al. (1993), Denker and Koch (2002), Giuliano-Antonini and Weber (2011) and Weber (2011). As pointed out in Weber (2011, Remark 4.1), the argument in Denker and Koch (2002) needs some complementary explanations: for the evolution of the ASLLT the reader is referred to Denker and Koch (2002) and Weber (2011, Section 4).

In this work we address problem 4 of Denker and Koch (2002) and prove an almost sure local limit theorem for uniformly recurrent Markov chains in the case where the values of S_n are all of the form $na + kd$, $k \in \mathbb{Z}$, with d being the maximal span. This question was raised by Denker and Koch (2002, p. 149, lines −2, −1). Our result in particular contains the case of finite state Markov chains with all strictly positive transitions between states. For example let $\{\xi_k\}$ be the 0–1 state Markov chain generated by a 2×2 matrix $\mathbf{P}$, where

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22}, \end{pmatrix}$$

$p_{ij} > 0$, $i, j = 1, 2$. By the analogy to the i.i.d. case let us call $\{\xi_k\}$ Markov trials. Set

$$\gamma = 1 - p_{12} - p_{21}, \qquad \pi_0 = \frac{p_{21}}{p_{12}+p_{21}}, \qquad \pi_1 = \frac{p_{12}}{p_{12}+p_{21}}.$$

Consider $\{f(\xi_k)\}$ where $f(0) = -\pi_1, f(1) = \pi_0$. It is not difficult to see (cf. Szewczak, 2012, p. 1206) that the asymptotic (or spectral) variance σ^2 of $\{f(\xi_k)\}$ satisfies

$$\sigma^2 = E_\pi(f^2(\xi_0)) + 2\sum_{n\geq 1}E_\pi(f(\xi_0)f(\xi_n)) = \pi_0\pi_1\left(1 + \frac{2\gamma}{1-\gamma}\right) = \pi_0\pi_1\frac{1+\gamma}{1-\gamma}.$$

It turns out that for Markov trials the following corresponds to Corollary 1 in Denker and Koch (2002):

$$\frac{1}{\ln n}\sum_{\nu=1}^{n}\frac{\sigma}{\sqrt{\nu}}I_{[S_\nu=\kappa_\nu]} \overset{\text{a.s.}}{\to} \mathfrak{n}(\kappa) \quad \text{as } \frac{\kappa_\nu}{\sigma\sqrt{\nu}} \to_\nu \kappa, \tag{1.1}$$

where the κ_ν are of the form $-\nu\pi_1 + k$. The relation (1.1) is the immediate consequence of Theorem 1.

We say that $\{\xi_k\}$ is uniformly recurrent if the condition below holds: *Condition* (Ψ):

$$0 < \psi' = \inf_{y,x\in\mathbb{S}}\frac{P(\xi_1 = y \mid \xi_0 = x)}{P(\xi_1 = y)} \quad \text{and} \quad \sup_{y,x\in\mathbb{S}}\frac{P(\xi_1 = y \mid \xi_0 = x)}{P(\xi_1 = y)} = \psi^* < \infty.$$

From Corollary 22.11 on p. 381, Volume II, and Theorem 7.5, on p. 210, Volume I, in Bradley (2007), it follows that if $\{\xi_k\}$ is uniformly recurrent then it is at least exponentially fast ψ-mixing. For example this is the case when $\{\xi_k\}$ is driven by a stochastic matrix $\mathbf{P}$ with all strictly positive elements.

Our main result is the following statement.

Theorem 1. *Suppose $\{\xi_k\}$ is a uniformly recurrent strictly stationary Markov chain and f is a Borel function such that the distribution of $f(\xi_1)$ is concentrated on $a + kd$, $k \in \mathbb{Z}$, with d being the maximal span and $E|X_1|^3 < \infty$. Then*

$$\frac{1}{\ln n}\sum_{\nu=1}^{n}\frac{\sigma}{\sqrt{\nu}}I_{[S_\nu=\kappa_\nu]} \overset{\text{a.s.}}{\to} d\mathfrak{n}(\kappa) \quad \text{as } \frac{\kappa_\nu - a_\nu}{\sigma\sqrt{\nu}} \to_\nu \kappa,$$

where $S_\nu = \sum_{k=1}^{\nu} f(\xi_k)$, $\sigma^2 = \sum_{k\in\mathbb{Z}}\mathrm{Cov}(X_0X_k)$, $a_\nu = \nu E(X_1)$ and the κ_ν are of the form $\nu a + kd$.

The proof of Theorem 1 uses ideas from Giuliano-Antonini and Weber (2011) and Szewczak (2003). The key role in this proof is played by (see Lemma 2) Edgeworth expansion in the conditional, or more generally operator, form (cf. Szewczak, 2006, 2008a,b). The work is organized as follows: auxiliary results required for the proof of Theorem 1 in Section 3 are established in Section 2.