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Nonparametric estimation problem for a time-periodic signal in a periodic noise

D. Dehay, K. El Waled *

IRMAR, CNRS umr 6625, Université Rennes 2, France

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ABSTRACT

In this paper we construct a kernel estimator of a periodic signal when the observation follows the model $d\zeta_t = f(t)dt + \sigma(t)dW_t$, where $f, \sigma : \mathbb{R} \to \mathbb{R}$ are continuous periodic and $\{W_t, t \geq 0\}$ is a Brownian motion. We state its consistency as well as the asymptotic normality.

1. Introduction

We consider the following model of periodic signal disturbed by noise whose variance is periodic

$$d\zeta_t = f(t)dt + \sigma(t)dW_t, \quad t \geq 0, \tag{1}$$

where $f, \sigma : \mathbb{R} \mapsto \mathbb{R}$ are two continuous periodic functions with the same period P, and $W = \{W_t, t \geq 0\}$ is a standard Brownian motion defined over a complete probability space $(\Omega, \mathcal{F}, P)$. Here we focus on the estimation of the time periodic drift function $f(\cdot)$ when we observe a trajectory of process (1) along a time interval $[0, T]$ as T goes to infinity. More precisely we are going to construct an estimator of $f(\cdot)$ based on a periodic kernel. The diffusion function $\sigma(\cdot)$ and the period $P > 0$ are assumed to be known. Moreover to avoid the trivial case, we also assume that the diffusion function is not identically null.

The estimation of $\sigma^2(\cdot)$ is not a problem for continuous time observation: the value of $\sigma^2(t)$ can be evaluated exactly from an arbitrarily short interval of observation $[t, t+\epsilon]$ using properties of the Brownian motion. Indeed the quadratic variation process $\{[\zeta]_t, t \geq 0\}$ of the process $\{\zeta_t, t \geq 0\}$ is such that $d[\zeta]_t = \sigma^2(t)dt$ (Klebaner, 2006). So consider the partition of the interval $[t, t+\epsilon]$ into n equal intervals $\Delta_i^{(n)}, i = 1, \ldots, n$ and denote by $\Delta\zeta_j^{(n)}$ the increment of ζ_t on $\Delta_i^{(n)}$, then it is well known that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\Delta\zeta_j^{(n)})^2 = \int_t^{t+\epsilon} \sigma^2(u)\, du, \quad P\text{-a.e.}$$

Hence $\sigma^2(\cdot)$ can be exactly determined for any t when the length T of the interval of observation is greater than the period P. The problem of estimation of P is a parametric problem that will be the subject of another work.

* Corresponding author.
E-mail addresses: dominique.dehay@univ-rennes2.fr (D. Dehay), khalil.elwaled@gmail.com, khalil.elwaled@univ-rennes2.fr (K. El Waled).

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From the periodicity of $f(\cdot)$ and $\sigma(\cdot)$, and the fact that the Brownian motion has independent increments, we will see that this statistical problem is an i.i.d. (independent and identically distributed) estimation problem. We state that the rate of convergence of the kernel estimator is the same as the rate of convergence for nonparametric estimation of density in the case of i.i.d. observations.

As an application, let $\{\xi_t, t \geq 0\}$ be the time dependent geometric Brownian motion which verifies the following linear stochastic differential equation

$$d\xi_t = f(t)\xi_t dt + \sigma(t)\xi_t dW_t. \tag{2}$$

The connection between these processes is given by

$$d\zeta_t = \frac{d\xi_t}{\xi_t}.$$

So the observation of $\{\xi_t, t \in [0, T]\}$ is equivalent to the observation of $\{\zeta_t, t \in [0, T]\}$, and the estimation of the time drift component in model (2) is identical to this estimation in model (1). Equations of such a type arise in many domains for instance in finance (Karatzas and Shreve, 1991; Klebaner, 2006) (Black–Scholes–Merton model), mechanics (Has'minskiĭ, 1980; Jankunas and Khas'minskiĭ, 1997) and in biology (Collet and Martinez, 2008; Höpfner, 2007). Here we introduce a time periodic influence in the drift and the diffusion coefficients of the model. Models with periodic structure have created a large amount of interest (see e.g. Gardner et al. (2006), Has'minskiĭ (1980), Serpedin et al. (2005)) and recently, the parameter estimation problem for time-periodic inhomogeneous diffusion processes have been considered (see Dehay (submitted for publication), and Höpfner and Kutoyants (2010)).

The argument of the paper is as follows. In Section 2 we state that the process $\{\zeta_t, t \geq 0\}$ in model (1) can be represented with a functional autoregressive time series whose state space is $\mathcal{C}[0, P]$, the space of continuous functions on $[0, P]$. The process $\{\zeta_t, t \geq 0\}$ is an inhomogeneous Markov process which is recurrent when $F(P) = 0$, and transient otherwise. We see in the next sections that the properties of the estimator under consideration do not depend on the recurrent or transient property of the process. Then in Section 3 we consider an estimator of the function $f(\cdot)$ constructed with a periodic kernel from a trajectory of the process continuously observed in an interval $[0, T]$. Section 4 is devoted to the consistency and the asymptotic normality of the estimator as $T \to \infty$. We also study the rate of convergence to 0 of the mean square error. In Section 5 we state the strong consistency of the estimator in the particular case of the triangular kernel.

Besides for simplicity of presentation we also assume that the initial value ζ_0 is constant and equal to 0.

2. Properties of the observation $\{\zeta_t, t \geq 0\}$

The process $\{\zeta_t, t \geq 0\}$ is a Gaussian process with independent increments, the mean of ζ_t being $F(t) := \int_0^t f(u)\, du$ and its variance $\int_0^t \sigma^2(u)\, du$. Moreover we have

$$\zeta_{nP+t} = nF(P) + F(t) + \sum_{k=0}^{n-1} Z_k + \mathbf{Z}_n(t) \tag{3}$$

for all $n \in \mathbb{N}$ and $t \in [0, P]$, where $\mathbf{Z}_k(t) := \int_0^t \sigma(u)\, dW_u^{(kP)}$, $Z_k := \mathbf{Z}_k(P)$ and $W_u^{(kP)} := W_{kP+u} - W_{kP}$. The Brownian motions $\{W_u^{(kP)}, u \in [0, P]\}$, $k \geq 0$, as well as the processes $\mathbf{Z}_k := \{\mathbf{Z}_k(u), u \in [0, P]\}$, $k \geq 0$, are independent and identically distributed in $\mathcal{C}[0, P]$. Thus the process $\{\zeta_t, t \geq 0\}$ is an inhomogeneous Markov process with a periodic transition semigroup. To get a better insight on the structure of this process, following Höpfner and Kutoyants (2010) define the P-segments time series

$$\mathbf{Y}_n := \{\zeta_{nP+t}, t \in [0, P]\}, \quad n \in \mathbb{N}.$$

Thanks to decomposition (3) this time series fulfills a functional autoregressive representation

$$\mathbf{Y}_n = \mathbf{Y}_{n-1}(P) + F(\cdot) + \mathbf{Z}_n$$

and $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ is a homogeneous Markov sequence with state space $\mathcal{C}[0, P]$. From the fact that the real-valued random variables $Z_k = \mathbf{Z}_k(P)$, $k \in \mathbb{N}$ are i.i.d., the strong law of large numbers applies and we easily obtain that

$$\lim_{n\to\infty} \sup_{t\in[0,P]} \left| \frac{1}{n}\mathbf{Y}_n(t) - F(P) \right| = 0 \quad P\text{-a.e.}$$

Then we deduce the following P-a.e. limits

$$\lim_{t\to\infty} \zeta_t = \begin{cases} -\infty & \text{if } F(P) < 0 \\ \infty & \text{if } F(P) > 0. \end{cases}$$

When $F(P) = 0$, the Hartman Wintner law of iterated logarithm applies and we obtain that

$$P\left[\liminf_{n\to\infty} \frac{\zeta_{nP}}{\sqrt{2n \ln\ln n}} = -\sqrt{G(P)}\right] = P\left[\limsup_{n\to\infty} \frac{\zeta_{nP}}{\sqrt{2n \ln\ln n}} = \sqrt{G(P)}\right] = 1$$