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Some stochastic comparisons in series systems with active redundancy

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ABSTRACT

We compare the lifetimes of series systems with different allocations of active redundancy using a variety of stochastic comparisons. It is assumed that only one spare can be allocated to each component of the system.

1. Introduction

One way to increase the reliability of a system is the use of redundant components. In general, there are three kinds of redundancies, i.e. *active* (*hot*), *passive* (*cold*) and *warm* redundancy. Active redundancy means that the redundant component undergoes the regular stress level, that is, the same stress level as when it is the principal component. In a system with active redundancy the principal component and the redundant one form a parallel system. In the state of passive redundancy the redundant component has zero failure rate and then it cannot fail while it remains in this state. Warm redundancy is an intermediate case. The component in a warm state operates under a milder stress level than when it is the principal component and at the failure of the principal component it will immediately operate under the regular stress level. Recently Cha et al. (2008) proposed a general standby system that includes the cases of cold, hot and warm standby.

In this paper we consider series systems with active redundancies. We study the problem of where to allocate the redundancies in order to maximize, in different senses of stochastic comparison, the lifetime of the systems.

The problem of where to allocate redundancies in a system to obtain optimal configurations has been studied using a variety of stochastic comparisons. There is a great amount of work on the study of redundancy allocation using stochastic orders. Some remarkable works are Boland et al. (1988, 1989, 1994), Meng (1996) and Singh and Singh (1997). Extensive references on stochastic orders are Shaked and Shanthikumar (2007) and Müller and Stoyan (2002). The former of these books summarizes many works on redundancy allocation and contains a large list of references therein.

Stochastic comparisons may be also made through what in Boland et al. (2004) is called the stochastic precedence order. Some other works where this type of stochastic comparison is used are Blyth (1972), Li and Hu (2008), Romera et al. (2004) and Singh and Misra (1994).

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We interpret all random variables as lifetimes of components or systems; therefore we will consider nonnegative random variables. We will assume that all distribution functions $B(t)$ satisfy $B(0) = 0$. We use the notation $\bar{B} \equiv 1 - B$. The maximum and minimum will be denoted as $\vee$ and $\wedge$, respectively. The terms increasing and decreasing will be used in the non-strict sense.

Let X and Y be random variables with corresponding distribution functions $F(t)$ and $G(t)$. If X and Y are absolutely continuous, let us denote by $\lambda(t)$ and $\mu(t)$ their respective hazard rate functions and by $r(t)$ and $q(t)$ their corresponding reversed hazard rate functions.

The following definitions of stochastic comparisons between two random variables will be needed ($a/0$ will be taken equal to ∞ whenever $a > 0$). X is said to be smaller than Y in the:

1. *Usual stochastic order* (denoted as $X \leq_{st} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all real t. We will write $X =_{st} Y$ if $F(t) = G(t)$ for all t.
2. *Hazard rate order* (denoted as $X \leq_{hr} Y$) if $\bar{G}(t)/\bar{F}(t)$ increases in $t \geq 0$. If X and Y are absolutely continuous, then $X \leq_{hr} Y$ is equivalent to $\lambda(t) \geq \mu(t)$ for all $t \geq 0$.
3. *Reversed hazard rate order* (denoted as $X \leq_{rh} Y$) if $G(t)/F(t)$ increases in $t > 0$. If X and Y are absolutely continuous, then $X \leq_{rh} Y$ is equivalent to $r(t) \leq q(t)$ for all $t \geq 0$.
4. *Increasing concave order* (denoted as $X \leq_{icv} Y$) if $\int_0^t \bar{F}(x)\mathrm{d}x \leq \int_0^t \bar{G}(x)\mathrm{d}x$ for all $t \geq 0$.
5. *Probabilistic relation* (denoted as $X \leq_{pr} Y$) if $P(X > Y) \leq P(Y > X)$.

The relation $\leq_{pr}$ in Boland et al. (2004) is called the stochastic precedence order. This relation is not a partial order since it does not meet the transitive property requirement (see Blyth (1972)). Unlike the stochastic orders that we consider in this paper, which only depend on the marginal distributions of X and Y, the probabilistic relation depends on the joint distribution of these random variables. Therefore, the relation $\leq_{pr}$ may be of special interest when comparing X and Y because it does take into account the possible dependence between these random variables. The relation $X \leq_{st} Y$ in general does not imply $X \leq_{pr} Y$; nevertheless if X and Y are independent this implication holds (see Boland et al. (2004)).

$X \leq_{hr} Y$ and also $X \leq_{rh} Y$ imply $X \leq_{st} Y$ and this in turn implies $X \leq_{icv} Y$. Further details about these stochastic orders may be found in Shaked and Shanthikumar (2007).

Throughout the paper we implicitly assume that the lifetimes $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n$ are independent. We denote the distribution function of the lifetime X_i (Y_i) by $F_i(t)$ ($G_i(t)$). If X_i (Y_i) is absolutely continuous then we will denote its probability density function by $f_i(t)$ ($g_i(t)$), $i = 1, 2, \dots, n$.

In Section 2 of this paper we analyze the allocation of one spare in a series system. In Section 3 we study the allocation of more than one spare.

2. Allocation of one spare

Consider a series system with n components. Suppose that there are two spares R_1 and R_2 to be allocated as active redundancies to the components C_1 and C_2, respectively, but only one of the spares can be allocated. Let $X_1, X_2, \dots, X_n$ be the lifetimes of components. Let X_1 and X_2 be the lifetimes of components C_1 and C_2 and Y_1 and Y_2 be the lifetimes of spares R_1 and R_2, respectively. Let $Z = \wedge(X_3, X_4, \dots, X_n)$ and denote by $H(t)$ the distribution function of Z. Then

$$U_1 = \wedge[\vee(X_1, Y_1), X_2, Z] \quad \text{and} \quad U_2 = \wedge[X_1, \vee(X_2, Y_2), Z]$$

represent the lifetimes of the two possible configurations of the system.

Proposition 1. *Suppose that either of the following conditions holds:*

(a) $X_1 \leq_{icv} X_2$ *and* $Y_1 \geq_{st} Y_2$,
(b) $X_1 \leq_{icv} X_2$, $Y_2 \leq_{st} X_2$ *and* $X_1 \leq_{st} Y_1$.

Then $U_1 \geq_{icv} U_2$.

Proof. Note that

$$P(U_1 > t) = \left[\bar{F}_2(t) - F_1(t)G_1(t) + G_1(t)F_1(t)F_2(t)\right]\bar{H}(t)$$

and

$$P(U_2 > t) = \left[\bar{F}_1(t) - F_2(t)G_2(t) + G_2(t)F_1(t)F_2(t)\right]\bar{H}(t).$$

Thus,

$$P(U_1 > t) - P(U_2 > t) = [F_1(t)\bar{F}_2(t)\bar{G}_1(t) - F_2(t)\bar{F}_1(t)\bar{G}_2(t)]\bar{H}(t).$$

Suppose that condition (a) holds. Using that $Y_1 \geq_{st} Y_2$ we obtain

$$P(U_1 > t) - P(U_2 > t) \geq (\bar{F}_2(t) - \bar{F}_1(t))\bar{G}_2(t)\bar{H}(t).$$