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Examples of optimal prediction in the infinite horizon case

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ABSTRACT

Timing of financial decisions, especially in a volatile market such as the one we are in now, is crucial to maintaining and growing wealth. Without premonition or inside information, buyers and sellers of financial assets may experience a form of remorse for selling too late or buying too early. A valuable tool in reducing such regret is an algorithm that tells the asset holder when to sell "optimally". In the case of a Brownian-valued asset, Graversen et al. (2000) proposed the strategy of *Optimal Prediction*, where the Brownian motion is stopped as close as possible, in the mean-square sense, to its ultimate maximum over the entire term $[0, T]$. Any candidate for the optimal stopping time must be adapted to the filtration of the underlying asset, since no inside information is to be assumed. Later work, nicely summarized in the book of Peskir and Shiryaev (2006), has extended this field to include different non-adapted functionals and different measures of "closeness". In this article, we seek to extend the field of optimal prediction to the perpetual, or infinite horizon, case. Some examples related to the ultimate risk associated with holding a toxic liability and the ultimately best time to sell a stock are presented, and their closed form solutions are computed.

1. History

In volatile economic times, the timing of purchases, locking in of mortgages, and the sale of part or all of a portfolio takes on a critical importance. The ability to quantify the effect of carrying out such an action, having done so too late or too soon, is also critical to long term financial planning. This is true whether the entity carrying out such a decision is an individual or a large corporation. Of course, we can only predict when the optimal time to carry out an action is; it is revealed to us only in the future when it may be late.

We are left with searching for other methods to minimize our remorse. Options are certainly one method of doing so, exchanging a premium up front for a guaranteed payout in the case of an agreed-upon event "in the future". The Russian option, designed by Shepp and Shiryaev (1993), fairly prices the financial instrument that returns the time-discounted value of the maximum of an asset over an agreed upon term.

However, the recent economic crisis has shown us that even large banks and insurance companies are not immune to defaulting on the payoff of an option. An alternative would be useful. One such alternative has been proposed in the seminal paper of Graversen et al. (2000) and extended in the work of du Toit and Peskir (2007). This work suggests that by tracking the right statistics of an asset, one can time an action that minimizes the average $\mathcal{L}^2$ distance between the ultimately highest value of an asset and the stopped value of the asset when the action is carried out. Urusov (2002) has shown this to be equivalent to minimizing the absolute difference between the time the action is carried out and the time of the ultimate maximum. Keep in mind that no inside information is assumed or allowed — all stopping times are adapted to the filtration generated by the asset itself.

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This field has come to be known as *Optimal Prediction* (see e.g. du Toit and Peskir (2007) and references therein.) It is an exciting new subset of Optimal Stopping problems, where the payoff function is not yet known when the decision to stop is made. So far, the problems posed and solved have concentrated on the finite horizon case. In this work, we seek to extend this notion to the case of infinite horizon, and show some areas where there are vital differences between the finite and infinite horizon setting. We also note here that Optimal Stopping problems have a rich history, and are also well studied in the discrete time formulation. For an excellent reference, please consult Chow et al. (1971).

1.1. Original problem

We begin with the original problem posed in Graversen et al. (2000): Consider a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and a standard Brownian motion $\{W_t\}_{t\geq 0}$ that lives on this space. For every realized path $\omega \in \Omega$ of the Brownian motion, there are statistics generated associated with W. The authors are interested in $S_t := \max_{0\leq s\leq t} W_s$. Specifically, they wish to predict the value of S_1, but only with information generated by the path of W until the decision is made to stop. Of course, S_1 is a random variable that depends on the entire path of the Brownian motion until time 1, The metric for prediction chosen by the authors is the mean square distance of the stopped Brownian motion to S_1, its *ultimate maximum*.

Symbolically, this problem is written as

$$V_* = \inf_{0\leq\tau\leq 1} \mathbb{E}\left[(S_1 - W_\tau)^2 \mid W_0 = 0, S_0 = 0\right]. \tag{1.1}$$

Although the payoff functional in the statement of the problem above is not adapted to the filtration generated by the Brownian motion, the authors are successful in resolving this difficulty. They do so by formulating an equivalent one-dimensional optimal stopping problem, reducing the conditional expectation above to one that is dependent on the sufficient statistic

$$Z_t \equiv \frac{S_t - W_t}{\sqrt{1-t}} \tag{1.2}$$

where $Z_0 = 0$ in the specific case (1.1) above. With the notation

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ \Phi(x) &= \int_{-\infty}^{x} \phi(y)\,dy \end{aligned} \tag{1.3}$$

the optimal stopping time and minimum mean square distance to S_1 are found to be

$$\begin{aligned} \tau^* &:= \inf\{s \mid Z_s \geq z^*\} \\ V_* &= 2\Phi(z^*) - 1 \end{aligned} \tag{1.4}$$

where z^* the unique root of the equation

$$4\Phi(z^*) - 2z^*\Phi(z^*) - 3 = 0. \tag{1.5}$$

The elegance of this result cannot be overstated. The authors are able to transform a finite-horizon optimal stopping problem with a seemingly non-adapted payoff functional into an infinite horizon optimal stopping problem with an adapted payoff. By doing so, they also provide a closed form solution and optimal stopping rule. Unfortunately, as detailed in the next section, a closed form solution is not always possible in the general finite horizon case.

1.2. Extension

The natural extension of this problem, where the Brownian motion can possess a drift term, was solved in du Toit and Peskir (2007), although no closed form solution is presented. In this more complicated problem, the optimal stopping boundary is characterized as the unique solution to a nonlinear integral equation (or a system of two such equations if there are two optimal boundaries.) Both analytic and numerical techniques are employed by the authors in du Toit and Peskir (2007) to solve these integral equations. The work we present in this paper employs much simpler techniques to determine the optimal stopping time. Our goal is to extend the notion of Optimal Prediction to the infinite horizon case. Optimal stopping over an infinite horizon is very often the first example presented to beginning students. Closed form solutions can, in many cases, be computed and intuition about their finite horizon cousins can be obtained. Such functions can be useful as upper or lower bounds for finite horizon Optimal Prediction problems, as well as provide intuition as to whether solutions can be obtained at all. This is our motivation for the work we present in this paper, and care is taken to formulate reasonable problems that possess closed form solutions. To our knowledge, this is the first work to present an extension of Optimal Prediction to the infinite horizon setting.