Statistics and Probability Letters 80 (2010) 969–974

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Ordering properties of convolutions of heterogeneous Erlang and Pascal random variables

Peng Zhao [a,*], N. Balakrishnan [b,1]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

[b] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

ARTICLE INFO

Article history:
Received 19 August 2009
Received in revised form 6 February 2010
Accepted 10 February 2010
Available online 21 February 2010

MSC:
primary 60E15
secondary 60K10

Keywords:
Majorization
p-larger order
Reciprocal majorization order
Likelihood ratio order
Hazard rate order
Mean residual life order
Erlang convolution
Pascal convolution

ABSTRACT

In this paper, we study ordering properties of convolutions of heterogeneous Erlang and Pascal random variables in terms of the majorization order [*p*-larger order, reciprocal majorization order] of parameter vectors and the likelihood ratio order [hazard rate order, mean residual life order]. We establish, among other things, that weak majorization order [*p*-larger order, reciprocal majorization order] between scale parameter vectors and majorization order between shape parameter vectors imply likelihood ratio order [hazard rate order, mean residual life order] between convolutions of two heterogeneous Erlang or Pascal sets of variables. These results strengthen and generalize some of the results known from the literature.

1. Introduction

Convolutions of independent random variables often arise in a natural way in statistics, applied probability, life testing, and operations research. Since the distribution theory is quite complicated when the convolution involves independent and non-identical random variables, it is of great interest to investigate stochastic properties of convolutions and derive bounds and approximations on some characteristics of interest in this setup. Many results in this direction have appeared in the literature; see, for example, Boland et al. (1994), Sen and Balakrishnan (1999), Bon and Păltănea (1999), Kochar and Ma (1999), Korwar (2002), Khaledi and Kochar (2004), Mao et al. (in press), Zhao and Balakrishnan (2009a,b), Zhao et al. (2009), Kochar and Xu (2009) and Zhao and Hu (2010).

The gamma distribution has been widely applied in reliability, engineering and many other areas, and the negative binomial distribution is used commonly to model count data, especially in Bayesian inference. In this paper, we investigate ordering properties of convolutions of Erlang random variables (i.e., gamma random variables with integer-valued shape parameters) and Pascal random variables (i.e., negative binomial random variables with integer-valued shape parameters).

Here, we are concerned with non-negative random variables which are absolutely continuous or discrete with support on integers $N_0 = \{0, 1, \ldots\}$. For two absolutely continuous [discrete] random variables X and Y with their density [mass]

* Corresponding author.
E-mail addresses: zhaop07@gmail.com (P. Zhao), bala@mcmaster.ca (N. Balakrishnan).

[1] Visiting Professor at King Saud University (Saudi Arabia) and National Central University (Taiwan).

0167-7152/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2010.02.010

functions f_X $[p_X]$ and f_Y $[p_Y]$, the survival function $\bar{F}_X$ and $\bar{F}_Y$, X is said to be smaller than Y in the *likelihood ratio order*, denoted by $X \le_{\rm lr} Y$, if $f_Y(x)/f_X(x)$ $[p_Y(k)/p_X(k)]$ is increasing in x $[k \in N_0]$; X is said to be smaller than Y in the *hazard rate order*, denoted by $X \le_{\rm hr} Y$, if $\bar{F}_Y(x)/\bar{F}_X(x)$ $[\bar{F}_Y(k)/\bar{F}_X(k)]$ is increasing in x $[k \in N_0]$; X is said to be smaller than Y in the *stochastic order*, denoted by $X \le_{\rm st} Y$, if $\bar{F}_Y(x)$ $[\bar{F}_Y(k)] \ge \bar{F}_X(x)$ $[\bar{F}_X(k)]$ for all x $[k \in N_0]$; X is said to be smaller than Y in the *mean residual life order* (denoted by $X \le_{\rm mrl} Y$) if $\mathsf{E}X_t \le \mathsf{E}Y_t$, where $X_t = (X - t|X > t)$ is the residual life at age $t > 0$ of the random lifetime X. For a more comprehensive discussion on various stochastic orderings, see Shaked and Shanthikumar (2007) and Müller and Stoyan (2002).

The notion of *majorization* is quite useful in establishing various inequalities. Let $x_{(1)} \le \cdots \le x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$. For two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{x} \in \mathbb{R}^n$ is said to *majorize* $\mathbf{y}$ (written as $\mathbf{x} \overset{\rm m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$ and $\sum_{i=1}^j x_{(i)} \le \sum_{i=1}^j y_{(i)}$ for $j = 1, \ldots, n-1$; $\mathbf{x}$ is said to *weakly majorize* $\mathbf{y}$ (written as $\mathbf{x} \overset{\rm w}{\succeq} \mathbf{y}$) if $\sum_{i=1}^j x_{(i)} \le \sum_{i=1}^j y_{(i)}$ for $j = 1, \ldots, n$; $\mathbf{x} \in \mathbb{R}_+^n$ is said to be *p-larger* than $\mathbf{y} \in \mathbb{R}_+^n$ (written as $\mathbf{x} \overset{\rm p}{\succeq} \mathbf{y}$) if $\prod_{i=1}^j x_{(i)} \le \prod_{i=1}^j y_{(i)}$ for $j = 1, \ldots, n$; $\mathbf{x} \in \mathbb{R}_+^n$ is said to *reciprocal majorize* $\mathbf{y} \in \mathbb{R}_+^n$ (written as $\mathbf{x} \overset{\rm rm}{\succeq} \mathbf{y}$) if $\sum_{i=1}^j \frac{1}{x_{(i)}} \ge \sum_{i=1}^j \frac{1}{y_{(i)}}$ for $j = 1, \ldots, n$. From Kochar and Xu (2009), the following implications:

$$\mathbf{x} \overset{\rm w}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\rm p}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\rm rm}{\succeq} \mathbf{y}$$

hold for any two non-negative vectors $\mathbf{x}$ and $\mathbf{y}$. For more details on majorization, p-larger, and reciprocal majorization orders and their applications, one may refer to Marshall and Olkin (1979), Bon and Păltănea (1999), Khaledi and Kochar (2002), and Zhao and Balakrishnan (2009a).

We establish, among other things, that weak majorization order [p-larger order, reciprocal majorization order] between scale parameter vectors and majorization order between shape parameter vectors imply likelihood ratio order [hazard rate order, mean residual life order] between convolutions of two heterogeneous Erlang or Pascal sets of variables. These results strengthen and generalize the exponential and geometric results of Boland et al. (1994), Bon and Păltănea (1999), Mi et al. (2008), Zhao and Balakrishnan (2009a,b), Zhao and Hu (2010) and Mao et al. (in press).

2. Convolutions of Erlang random variables

Assume that $X_{(r,\lambda)}$ is a gamma random variable with density function

$$f_{(r,\lambda)}(t) = \frac{\lambda^r t^{r-1}}{\Gamma(r)} e^{-\lambda t}, \quad t \ge 0,$$

where $r > 0$ and $\lambda > 0$. Then, $X_{(r,\lambda)}$ becomes an Erlang random variable if r is an integer. In this section, we present some results on likelihood ratio, hazard rate and mean residual life orders based on convolutions of heterogeneous Erlang random variables.

2.1. Likelihood ratio ordering

We begin with several auxiliary results which are needed for establishing the main result.

Lemma 2.1. *For non-negative real numbers $\lambda_1 \le \cdots \le \lambda_n$ and integers $r_1 \ge \cdots \ge r_n$, we have the following equality:*

$$r_1\lambda_1 + \cdots + r_n\lambda_n = \sum_{i=1}^n (r_i - r_{i+1}) \sum_{j=1}^i \lambda_j,$$

where $r_{n+1} = 0$.

Proof. The result follows readily by writing

$$\begin{aligned} r_1\lambda_1 + \cdots + r_n\lambda_n &= \lambda_1[(r_1 - r_2) + (r_2 - r_3) + \cdots + (r_n - r_{n+1})] + \lambda_2[(r_2 - r_3) + (r_3 - r_4) + \cdots + (r_n - r_{n+1})] \\ &\quad + \cdots + \lambda_n(r_n - r_{n+1}). \quad \blacksquare \end{aligned}$$

Lemma 2.2. *Let $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1^*, \ldots, \lambda_n^*)$ be two non-negative vectors such that $(\lambda_1, \ldots, \lambda_n) \overset{\rm w}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*)$ with $\lambda_1 \le \cdots \le \lambda_n$ and $\lambda_1^* \le \cdots \le \lambda_n^*$. If integers $r_1, \ldots, r_n$ are such that $r_1 \ge \cdots \ge r_n$, then*

$$\Bigg(\underbrace{\lambda_1, \ldots, \lambda_1}_{r_1}, \ldots, \underbrace{\lambda_n, \ldots, \lambda_n}_{r_n}\Bigg) \overset{\rm w}{\succeq} \Bigg(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{r_1}, \ldots, \underbrace{\lambda_n^*, \ldots, \lambda_n^*}_{r_n}\Bigg). \tag{1}$$

Proof. According to the definition of weak majorization order and the conditions $\lambda_1 \le \cdots \le \lambda_n$ and $\lambda_1^* \le \cdots \le \lambda_n^*$, we have