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Functional limit theorems for linear processes in the domain of attraction of stable laws

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ABSTRACT

We study functional limit theorems for linear type processes with short memory under the assumption that the innovations are dependent identically distributed random variables with infinite variance and in the domain of attraction of stable laws.

1. Introduction

We consider the linear process $\{Z_j : j \in \mathbb{Z}\}$ defined by

$$Z_j = \sum_{k=-\infty}^{\infty} a_k \xi_{j-k}, \qquad (1)$$

where the innovations $\{\xi_j : j \in \mathbb{Z}\}$ are identically distributed random variables with infinite variance and the sequence of constants $\{a_k : k \in \mathbb{Z}\}$ is such that $\sum_{k\in\mathbb{Z}} |a_k| < \infty$. This case is referred to as short memory, or as short range dependence. The functional central limit theorem (FCLT) for the partial sums of the linear process, properly normalized, merely follows from the corresponding FCLT for the innovations being in the domain of attraction of the normal law; see Peligrad and Utev (2006). Then the limiting process has continuous sample paths and choosing the right topology in the Skorohod space $\mathbb{D}[0, 1]$ is not problematic. However, as shown by Avram and Taqqu (1992), the weak convergence of the partial sums of the linear process with independent innovations (i.i.d. case) in the domain of attraction of non-normal laws is impossible in the Skorohod J_1 topology on $\mathbb{D}[0, 1]$, but the functional limit theorem might still hold, under additional assumptions, in the weaker Skorohod M_1 topology (see Skorohod, 1956). Avram and Taqqu (1992) use the standard approach through tightness plus convergence of finite dimensional distributions. Here, we use approximation techniques and study weak convergence in $\mathbb{D}[0, \infty)$, i.e. the space of functions on $[0, \infty)$ that have finite left-hand limits and are continuous from the right. Given processes X_n, X with sample paths in $\mathbb{D}[0, \infty)$, we will denote by $X_n(t) \Longrightarrow X(t)$ the weak convergence in $\mathbb{D}[0, \infty)$ with one of the Skorohod topologies J_1 or M_1, and write $\xrightarrow{J_1}$ or $\xrightarrow{M_1}$, if the indicated topology is used. Note that if the limiting process X has continuous sample paths then $\Longrightarrow$ in $\mathbb{D}[0, \infty)$ with one of the Skorohod topologies is equivalent to weak convergence

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in $\mathbb{D}[0,\infty)$ with the local uniform topology. For definitions and properties of the topologies we refer to Jacod and Shiryaev (2003) and Whitt (2002).

To motivate our approach we first consider the linear process $\{Z_j : j \in \mathbb{Z}\}$ as in (1), where $\{\xi_j : j \in \mathbb{Z}\}$ is a sequence of i.i.d. random variables. There exist sequences $b_n > 0$ and c_n such that the partial sum processes of the i.i.d. sequence $\{\xi_j : j \in \mathbb{Z}\}$ converge weakly to an α-stable Lévy process X with $0 < \alpha < 2$ (see, e.g. Resnick, 1986, Proposition 3.4)

$$\frac{1}{b_n}\sum_{j=1}^{[nt]}(\xi_j - c_n) \overset{J_1}{\Longrightarrow} X(t) \tag{2}$$

if and only if there is convergence in distribution

$$\frac{1}{b_n}\sum_{j=1}^{n}(\xi_j - c_n) \overset{d}{\longrightarrow} X(1) \quad \text{in } \mathbb{R},$$

or, equivalently, there exist $p \in [0, 1]$ and a slowly varying function L, i.e., $L(sx)/L(x) \to 1$ as $x \to \infty$ for every $s > 0$, such that

$$\lim_{x\to\infty}\frac{\mathbb{P}(\xi_1 > x)}{\mathbb{P}(|\xi_1| > x)} = p \quad \text{and} \quad \lim_{x\to\infty}\frac{\mathbb{P}(|\xi_1| > x)}{x^{-\alpha}L(x)} = 1; \tag{3}$$

in that case the sequences $\{b_n, c_n : n \in \mathbb{N}\}$ in (2) can be chosen as

$$b_n = \inf\left\{x : \mathbb{P}(|\xi_1| \le x) \ge 1 - \frac{1}{n}\right\} \quad \text{and} \quad c_n = \mathbb{E}(\xi_1 I(|\xi_1| \le b_n)). \tag{4}$$

We have $b_n \to \infty$, $nP(|\xi_1| > b_n) \to 1$, and $nb_n^{-\alpha}L(n) \to 1$, as $n \to \infty$, by (3). We refer to Feller (1971) for α-stable random variables and their domains of attraction. If $\alpha = 2$, condition (2) holds if and only if the function $x \mapsto \mathbb{E}(\xi_1^2 I(|\xi_1| \le x))$ is slowly varying; in that case the sequence b_n can be chosen as satisfying $nb_n^{-2}\mathbb{E}(\xi_1^2 I(|\xi_1| \le b_n)) \to 1$ as $n \to \infty$, the c_n as in (4), and X is a Brownian motion. In any case, (2) implies that the function $x \mapsto \mathbb{E}(\xi_1^2 I(|\xi_1| \le x))$ is regularly varying with index $2-\alpha$, that is, there exists a slowly varying function ℓ such that (if $\alpha < 2$ then $\ell(x) = \alpha L(x)/(2-\alpha)$ where L is as in (3))

$$\lim_{x\to\infty}\frac{\mathbb{E}(\xi_1^2 I(|\xi_1| \le x))}{x^{2-\alpha}\ell(x)} = 1. \tag{5}$$

Astrauskas (1983) and Davis and Resnick (1985) show that if the coefficients $\{a_k : k \in \mathbb{Z}\}$ are such that

$$\sum_{k=-\infty}^{\infty}|a_k|^r < \infty \quad \text{for some } r < \alpha,\ 0 < r \le 1, \tag{6}$$

and if (2) holds then the linear process $\{Z_j : j \in \mathbb{Z}\}$ defined by (1) satisfies

$$\frac{1}{b_n}\sum_{j=1}^{n}\left(Z_j - Ac_n\right) \overset{d}{\longrightarrow} AX(1) \quad \text{in } \mathbb{R}, \text{ where } A = \sum_{k\in\mathbb{Z}} a_k.$$

For the case $\alpha \in (0, 2)$, Avram and Taqqu (1992) show that if $a_k \ge 0$, $k \in \mathbb{Z}$, satisfy (6) and if additional constraints are imposed for $\alpha \ge 1$ (see Avram and Taqqu, 1992, Theorem 2), then (2) implies

$$\frac{1}{b_n}\sum_{j=1}^{[nt]}(Z_j - Ac_n) \overset{M_1}{\Longrightarrow} AX(t) \tag{7}$$

and that the convergence in (7) is impossible in the J_1-topology. We show in Corollary 1 that no additional assumptions are needed for $\alpha \ge 1$. It is still not known to what extent one can relax the condition that all a_k have the same sign to get convergence in (7) with any topology weaker than J_1. With our approach we reduce this problem to continuity properties of addition in a given topology (see Section 3). When $\alpha = 2$ then (7) holds with any real constants a_k satisfying (6) (see, e.g. Peligrad and Utev, 2006; Moon, 2008, and the references therein).

We now consider identically distributed, possibly dependent, random variables $\{\xi_j : j \in \mathbb{Z}\}$ with $\mathbb{E}\xi_1^2 = \infty$. Note that if (5) holds with $\alpha \in (0, 2]$ then $\mathbb{E}|\xi_1|^\beta < \infty$ for every $\beta \in (0, \alpha)$, thus condition (6) ensures that each Z_j in (1) is a.s. converging series, since

$$\mathbb{E}|Z_j|^r \le \sum_{k\in\mathbb{Z}}|a_k|^r\mathbb{E}|\xi_{j-k}|^r = \mathbb{E}|\xi_1|^r\sum_{k\in\mathbb{Z}}|a_k|^r < \infty.$$

Our main result is the following.

Theorem 1. *Let a linear process $\{Z_j : j \in \mathbb{Z}\}$ be defined by (1), where $\{\xi_j : j \in \mathbb{Z}\}$ and $\{a_k : k \in \mathbb{Z}\}$ satisfy (5) and (6) with $\alpha \in (0, 2]$. Assume that $\{b_n, c_n : n \in \mathbb{N}\}$ are sequences satisfying the following conditions:*

$$b_n \to \infty, \quad \frac{c_n}{b_n} \to 0, \quad \textit{and} \quad \limsup_{n\to\infty} nb_n^{-2}E(\xi_1^2 I(|\xi_1| \le b_n)) < \infty, \tag{8}$$