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On generalized correlation functions of intrinsically stationary processes of order k

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ABSTRACT

In the first part of the paper we prove some inequalities for the functions mentioned in the title. In the second part we investigate definitizable functions defined on an interval $(-a, a)$ and show that they can be extended to generalized correlation functions on $\mathbb{R}$.

1. Introduction

By a second-order random process Z on $\mathbb{R}$ we mean a mapping

$$Z : \mathbb{R} \mapsto L_2(\Omega, \mathcal{A}, P)$$

where $(\Omega, \mathcal{A}, P)$ is a probability space. We denote by $(\cdot, \cdot)$ the inner product of the Hilbert space $L_2(\Omega, \mathcal{A}, P)$, the symbol $\|\cdot\|$ denotes the corresponding norm and $\mathbb{E}$ stands for expectation, i.e.,

$$(X, Y) = \int_\Omega X \cdot \overline{Y}\,\mathrm{d}P = \mathbb{E}\,(X \cdot \overline{Y}) \quad \text{and} \quad \|X\| = \sqrt{(X, X)}.$$

A second-order process Z is called *continuous* if

$$\lim_{x \to x_0} \|Z(x) - Z(x_0)\| = 0$$

holds for all $x_0 \in \mathbb{R}$. The process Z is said to be *second-order stationary*, or simply *stationary*, if $\mathbb{E}\,(Z(x))$ does not depend on x and $\mathbb{E}\,(Z(x) \cdot \overline{Z(y)})$ is a function of $x - y$:

$$(Z(x), Z(y)) = C(x - y), \quad x, y \in \mathbb{R}.$$

The function C is called the *correlation function* of Z. It is well known that correlation functions are *positive definite*, i.e., the inequality

$$\sum_{i,j=1}^{n} C(x_i - x_j) c_i \overline{c_j} \geq 0$$

holds for an arbitrary choice of the positive integer n, complex numbers $c_j \in \mathbb{C}$ and $x_j \in \mathbb{R}$.

For an arbitrary set G we denote by $\mathcal{M}_f(G)$ the set of all finitely supported complex measures on G. Each element μ of $\mathcal{M}_f(G)$ can be written as

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$$\mu = \sum_{i=1}^{n} c_i \delta_{x_i}$$

where $c_i \in \mathbb{C}$ and δ_x denotes the one-point probability measure concentrated at x. If G is a commutative group we use additive notation and define the measure $\tilde{\mu}$ by

$$\tilde{\mu} = \sum_{i=1}^{n} \overline{c_i} \delta_{-x_i}$$

and the convolution $\mu * g$ by

$$\mu * g(t) = \int_G g(t-x)\, \mathrm{d}\mu(x) = \sum_{i=1}^{n} c_i g(t - x_i), \quad t \in G$$

where g is an arbitrary function on G with values in a Hilbert space. If $\nu = \sum_{j=1}^{m} d_i \delta_{y_i} \in \mathcal{M}_f(G)$, then

$$\mu * \nu = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j \delta_{x_i + y_j}.$$

Now let k be a positive integer. A second-order process Z on $\mathbb{R}$ is said to be *intrinsically stationary of order* k or a k-IRF if the process $x \mapsto \mu * Z(x)$ is stationary for all $\mu \in \mathcal{M}_f(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} x^j \, \mathrm{d}\mu(x) = 0, \quad j = 0, \ldots, k. \tag{1}$$

We refer to Sasvári (submitted for publication) for a short historical survey on the concept of intrinsical stationarity.[1] The book of Chilès and Delfiner (1999) contains geostatistical applications. A complex-valued function K on $\mathbb{R}$ is called a *generalized correlation* of a k-IRF Z if it is *hermitian*, i.e., $K(-x) = \overline{K(x)}$, and

$$(\mu * Z(x), \nu * Z(y)) = \mu * \tilde{\nu} * K(x - y)$$

holds for all $x, y \in G$ and for all μ and ν in $\mathcal{M}_f(\mathbb{R})$ satisfying (1). Matheron (1973) proved that every continuous k-IRF admits a generalized correlation. A more general existence result can be found in Sasvári (submitted for publication). It follows immediately from the definition of K that the function $\mu * \tilde{\mu} * K$ is positive definite for all $\mu \in \mathcal{M}_f(\mathbb{R})$ satisfying (1). Especially,

$$\mu * \tilde{\mu} * K(0) \geq 0$$

for such μ, i.e., K is *conditionally positive definite of order* k in the sense of Matheron (1973, page 450). As shown by Matheron (1973), for every continuous, conditionally positive definite function K of order k there exists a k-IRF Z such that K is a generalized correlation of Z.

Instead of conditionally positive definite functions in the above sense we will work with a special case of definitizable functions (cf. Sasvári (1994), Chapter 6) which are defined as follows. Let G be an arbitrary commutative group[2] and let $\mathbf{1}$ be the function defined by $\mathbf{1}(x) = 1, x \in G$. For a positive integer k we denote by $P(\mathbf{1}, k, G)$ the set of all complex-valued hermitian functions f on G such that the function $f * \mu * \tilde{\mu}$ is positive definite for all $\mu \in \mathcal{M}_f(G)$ of the form

$$\mu = \mu_1 * \cdots * \mu_k \tag{2}$$

where $\mu_j \in \mathcal{M}_f(G)$ and $\int_G \mathbf{1}\, \mathrm{d}\mu_j = 0$. By $P(\mathbf{1}, 0, G)$ we denote the set of all positive definite functions on G. If G is a topological group then $P^c(\mathbf{1}, k, G)$ denotes the set of continuous functions in $P(\mathbf{1}, k, G)$. Note that a continuous function f on $\mathbb{R}$ is conditionally positive definite of order $k - 1$ ($k > 0$) if and only if $f \in P^c(\mathbf{1}, k, \mathbb{R})$ (see the introduction in Sasvári (submitted for publication)). Thus the set of generalized correlation functions of continuous $(k-1)$-IRFs on $\mathbb{R}$ is equal to $P^c(\mathbf{1}, k, \mathbb{R})$.

By a classical result of Krein (1940), continuous positive definite functions defined on a finite interval $(-a, a)$ can be extended to continuous positive definite functions on $\mathbb{R}$. In the last section of the paper we will generalize this result for functions in $P^c(\mathbf{1}, k, \mathbb{R})$. The proof requires an inequality which we treat in Section 3.

2. Notation and auxiliary formulas

Let f be a complex-valued function defined on the set $\mathbb{Z}$ of all integers and define the function Δf by

$$\Delta f(n) = f(n+1) - f(n), \quad n \in \mathbb{Z}.$$

[1] Note that a continuous second-order process is a k-IRF if and only if it has *stationary increments of order* $k + 1$ in the sense of A.M. Yaglom and M.S. Pinsker (see Yaglom and Pinsker (1953) or Yaglom (1987)).

[2] In the present note we need only the cases $G = \mathbb{R}^d$ and $G = \mathbb{Z}$ (the additive group of integers).