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Comparisons of concordance in additive models

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ABSTRACT

In this note we compare bivariate additive models with respect to their Pearson correlation coefficients, Kendall's τ concordance coefficients, and Blomqvist β medial correlation coefficients. The conditions that enable the comparisons involve variability stochastic orders such as the dispersive and the peakedness orders. Specifically we show that we can compare the Kendall's τ concordance coefficients of Cheriyan and Ramabhadran's bivariate gamma distributions, in spite of the fact that it is hard (and not necessary) to compute them.

1. Introduction

Let Z_1 and Z_2 be two random variables that a probabilist uses to approximately describe some real world situation. It is often desired that Z_1 and Z_2, on one hand not be independent, and on the other hand not be totally dependent. A common way of doing this sort of modelling is to introduce three independent random variables, X_1, X_2, and Y, and then model Z_1 and Z_2 by

$$Z_1 = g(X_1, Y) \quad \text{and} \quad Z_2 = g(X_2, Y), \tag{1.1}$$

where g is some bivariate function. In the setup (1.1), X_1 and X_2 indicate the "individuality" that is associated with Z_1 and Z_2, whereas Y indicates the factors that give rise to the partial dependence between Z_1 and Z_2.

In the setup (1.1), it is sometimes of importance to figure out the influence of Y on the strength of positive dependence between Z_1 and Z_2. That is, suppose that the dependence between the two random variables in (1.1) can be chosen to be modelled using Y yielding (Z_1, Z_2) as in (1.1), or that it can be chosen to be modelled using $\widetilde{Y}$ yielding $(\widetilde{Z}_1, \widetilde{Z}_2)$ as follows

$$\widetilde{Z}_1 = g(X_1, \widetilde{Y}) \quad \text{and} \quad \widetilde{Z}_2 = g(X_2, \widetilde{Y}).$$

The question that arises then is what conditions on Y and $\widetilde{Y}$ imply that (Z_1, Z_2) is "less positively dependent" than $(\widetilde{Z}_1, \widetilde{Z}_2)$.

For example, Li and Pellerey (2011) considered, among other things, the comparison of

$$(Z_1, Z_2) = (\min\{X_1, Y\}, \min\{X_2, Y\})$$

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and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (\min\{X_1, \widetilde{Y}\}, \min\{X_2, \widetilde{Y}\}).$$

They showed that if Y is larger than $\widetilde{Y}$ in the ordinary stochastic order, then $(\widetilde{Z}_1, \widetilde{Z}_2)$ is more positively dependent than (Z_1, Z_2) in the sense of the concordance order, that is, the copula of $(\widetilde{Z}_1, \widetilde{Z}_2)$ is greater than, or equal to, the copula of (Z_1, Z_2) over the whole unit square; see, for example, Nelsen (2006). Fang and Li (2011) considered the comparison of

$$(Z_1, Z_2) = (\max\{X_1, Y\}, \max\{X_2, Y\})$$

and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (\max\{X_1, \widetilde{Y}\}, \max\{X_2, \widetilde{Y}\}).$$

They showed that if Y is smaller than $\widetilde{Y}$ in the ordinary stochastic order, then $(\widetilde{Z}_1, \widetilde{Z}_2)$ is more positively dependent than (Z_1, Z_2) in the same sense that was described above.

The purpose of this note is to compare

$$(Z_1, Z_2) = (X_1 + Y, X_2 + Y) \tag{1.2}$$

and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (X_1 + \widetilde{Y}, X_2 + \widetilde{Y}) \tag{1.3}$$

in a sense of positive dependence. More explicitly, we find conditions on Y and $\widetilde{Y}$ that yield a stronger positive dependence between $\widetilde{Z}_1$ and $\widetilde{Z}_2$, than between Z_1 and Z_2.

Random vectors of the form (1.2) have been used in the literature to model a variety of applications. Here is a sample of such usages:

- *Reliability theory*. The random vector in (1.2) can represent a replacement model similar to a model in Marshall and Shaked (1982, page 263). Specifically, in a reliability system that performs two tasks, Y is the lifetime of the original device that performs both tasks, and upon its failure, it is replaced by two devices with lifetimes X_1 and X_2, each of which performs only one of the tasks. Then (Z_1, Z_2) is the vector of the time periods of the performance of the two tasks.
- *Risk analysis.* Bauerle and Muller (1998, page 66) studied models of pairs of n-dimensional risky portfolios. In the case when $n = 1$, their model for (dependent) risks, that belong to a certain group, is $(g(X_1, Y), g(X_2, Y))$, for some bivariate function g, where X_1 and X_2 are the individual risk factors, and Y is the group-specific risk factor. Specifically, when $g(x, y) = x + y$, the model of Bauerle and Muller (1998) reduces to (1.2).
- *Combat target detection.* Youngren (1991) considered modelling the detection of an enemy unit that has some target elements such as a tank or a truck. When the unit has two elements, Youngren (1991, page 574) modelled the times to the detection of the elements by (1.2), where the random quantity Y captures the contribution of the common environmental factors on the time for detection of both elements, and the random quantity X_i captures the contribution of the other factors to the time of detection of element i, $i = 1, 2$.

Remark 1.1. At first glance it may not be clear what we may assume about Y and $\widetilde{Y}$ in (1.2) and (1.3) in order for $\widetilde{Z}_1$ and $\widetilde{Z}_2$ to be "more positively dependent" than Z_1 and Z_2. However, upon some reflection we may guess that if $\widetilde{Y}$ is "more variable" (or "more dispersed") than Y, then we may expect $\widetilde{Z}_1$ and $\widetilde{Z}_2$ to be "more positively dependent" than Z_1 and Z_2. The intuitive reason behind this is that the role of Y is to introduce the dependence between Z_1 and Z_2, and it does that by *adding* the same random quantity to both X_1 and X_2. Thus, the "more variable" Y is, the more it "forces" the sums $X_1 + Y$ and $X_2 + Y$ to vary, but to do it "together" and hence "be like each other", and as a result the "more dependent" Z_1 and Z_2 should be. Note that in the extreme case when Y is degenerate (that is, Y is "as small in variability as possible"), then Z_1 and Z_2 are independent. □

Verifying the intuition that is described in Remark 1.1, some of the results in this note are of the following form: If Y is smaller than $\widetilde{Y}$ in some variability sense, then (Z_1, Z_2) of (1.2) is smaller than $(\widetilde{Z}_1, \widetilde{Z}_2)$ of (1.3) with respect to some positive dependence sense.

Technically, we found the models in (1.2) and (1.3) to be quite complex for the purpose of comparing the copulas that are associated with (Z_1, Z_2) and with $(\widetilde{Z}_1, \widetilde{Z}_2)$. Thus our present study is more humble in the sense that we compare the Pearson product-moment correlation coefficients, the Kendall's τ concordance coefficients, and the Blomqvist's β medial correlation coefficients of (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ in (1.2) and (1.3).

It is worthwhile to mention that a comparison of the strength of dependence in the sense of SI (stochastic increasingness) of models that are similar to the ones in (1.2) and (1.3), but still quite different than these, is given in Proposition 3.1 of Khaledi and Kochar (2005).

In the next section we obtain results that compare (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ with respect to their Pearson product-moment correlation coefficients. These results are quite straightforward, but their importance is that they make up our first formalization of the intuition that is described in Remark 1.1. Our second formalization of that intuition is given in Section 3, where we develop comparisons of (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ with respect to their Kendall's τ concordance coefficients. A third formulation of the above intuition is described in Section 4, where we compare (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ with respect to their