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# Expected earnings of invested overflow strategies for $M/M/1$ queue with constrained workload

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ABSTRACT

In the case of $M/M/1$ queue with constrained workload (finite dam), we consider two strategies for investing overflow of the dam. Expected value of the present value of each strategy is computed using restricted Laplace transforms and properties of $M/M/1$ queues.



## 1. Introduction

In this paper, we consider an investment strategy for overflows of a dam, where the model for the amount of water in a dam is equivalent to the $M/M/1$ queue with constrained workload. By *constrained workload*, we mean uniformly bounded virtual waiting time with upper bound $V$. This process is also known as the *finite dam*. In dam theory, the busy period is called the wet period, and the constrained workload corresponds to a dam with finite capacity $V$. In the case of the $M/M/1$ model, the dam releases water at unit rate while receiving instantaneous rainfalls of exponential amounts, occurring according to a Poisson process. In queueing theory, this corresponds to a server which is only able to handle a workload of amount $V$, while accepting all service requests and providing service as long as the queue is not empty. We assume an initial workload (initial amount of water in dam) of amount $x$, where $0 < x \leq V$.

Suppose that the amount of overflow (i.e. the amount of workload exceeding $V$) corresponds to lost income. Of interest is a computation of the expected value of lost income, when the lost income is invested in a bank account with fixed interest rate $r$.

Another application of the problem at hand is found in gambling, in which the player plays a game repeatedly at unit intervals, each time paying a fixed price. The amount won after a single play of the game is assumed to have an exponential distribution. When the player's net earnings exceed a predetermined level $V$, he invests the excess into a bank account with a fixed interest rate.

*We propose and analyze two strategies for investing the amount of overflow.*

*Strategy* 1. First we consider the sum $W$ of invested lost income, where the sum is taken over all workload overflows until the end of the busy period. A potential application for the analogous dam model is the following. Each time the dam overflows,

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the overflow is used to produce energy, the earnings from which are in turn invested in an account with fixed interest rate $r$. We compute the sum of the invested earnings over the entire wet period. When viewed as a function of $V$ for fixed input and service rates, the sum $W$ of the invested earnings from energy sold can be maximized over $V$. Such analysis could help a utilities director determine the optimal water level of a dam or a service manager determine the optimal workload level.

In the gambling application, $W$ is equal to the sum of the amount of invested net earnings in excess of $V$, summed over each time the net earnings exceed $V$.

*Strategy* 2. The second strategy for summing the invested overflow when the a priori intent is to invest the amount of overflow a fixed number of times, say $n$, with the entire workload taken out with the $n$th overflow. If this case, we call the sum of invested workload $W_n$. If the total number of overflows $N$ in the wet period is less than $n$, then there will be lost opportunity. The investor offsets this risk by intending to take the entire workload out on the $n$th overflow.

A comparison of the expected value of $W$ with the expected value of $W_n$ for various values of $n$ or $V$ could help the investor (or dam engineer) manage the workload (or dam level and associated profits). We will express both strategies $W$ and $W_n$ in terms of the present value of money.

## 2. Background and finite dam model

Early work on dam theory (also known as storage systems) was done by Gani (1957), Moran (1959), Prabhu (1964), Takacs (1967) and Cohen (1969). Moran's original model is discrete, and it was later extended to a continuous model, as we consider in this work. See Prabhu (1964) or (1998) for an extensive summary of the work on storage theory.

The busy (wet) period of a queue (or dam) was considered early on by Phatarfod (1963, 1969), who found an *approximate* solution for the Laplace transform of the busy period of the finite dam and later a calculation of the first time to either emptiness or overflow in this same model. Heyman (1974) provided an approximation of the busy period of the $M/G/1$ queue (not finite), using a diffusion. Kinateder and Lee (2000) derived exact formulas for the Laplace transforms of the busy period and first time to emptiness or overflow using martingale methods, restricted Laplace transforms, and the Markov property, and separately using backward differential equations.

Let $Z_t$ denote the virtual waiting time of the $M/M/1$ queue with constrained workload as described above, with uniform bound (maximum workload) $V$.

In particular, for $t \geq 0$, let $A_t$ denote the arrival process, a Poisson process with intensity parameter $\nu > 0$, let $\xi_1, \xi_2, \ldots$ be independent inputs exponentially distributed with mean $\mu$ and let $X_t$ be defined by

$$X_t = \xi_0 + \Sigma_{i=1}^{A_t} \xi_i - t,$$

where $\xi_0$ has a probability distribution which is exponential with mean $\mu$ on $[0, V]$ and $V$ with probability $e^{\frac{-V}{\mu}}$. Let $Z_t$ denote the virtual waiting time of the $M/M/1$ queue with constrained workload; that is

$$Z_t = X_t - \max\left(0, \sup_{0<s<t} X_s - V\right)$$

with absorbing barrier at 0, where we ignore the initial idle period. Let $T_{0,V} = \inf\{t > 0 : X_t \notin (0, V]\}$ denote the first time to emptiness or overflow. Let the length of the busy period be denoted by $\tau = \inf\{t > 0 : Z_t = 0\}$.

## 3. Strategy definitions

In what follows, let $E_x$ denote expectation given $X_0 = x$ and $P_x$ likewise denote probability given $X_0 = x$.

In order to give a concrete definition of the functions that determine the aforementioned strategies, we must discuss *present value of money*, as our earnings functions will be expressed in terms of present value of money. Suppose an amount of money $m$ is invested in an interest bearing account at rate at time $t_a$ and held there until time $t_b$. At time $t_b$, the value of the investment is $me^{r(t_b - t_a)}$. The investor wants to have in hand the present value PV of that investment. A bank realizes that the investor could turn around and invest that amount PV into an account with interest rate $r$ for time 0 (present) until time $t_b$. Not wanting to lose any money, the bank would offer a PV of the investment such that $\text{PV}e^{rt_b} = me^{r(t_b - t_a)}$. Solve to get that $\text{PV} = me^{-rt_a}$ is the present value of $m$ when invested in an interest bearing account at rate at time $t_a$ and held there until time $t_b$. We will express all earnings in this paper in terms of present value.

Assume $\mu\nu \neq 1$. Let $N$ denote the number of overflows of the constrained workload $Z$ during the wet period, let $t_1, \ldots, t_N$ denote the times of the overflows, and let $Z^{(i)}$ denote the amount of overflow $i$. We note that nothing will be invested if there are no overflows ($N = 0$). Let $r$ be the continuously compounded interest rate. The first invested overflow strategy $W$ as described in Section 1 will yield the sum of each overflow, invested from the time of overflow until the end of the busy period; i.e. $(\Sigma_{i=1}^{N} Z^{(i)} e^{r(\tau - t_i)}) I_{\{N \geq 1\}}$. We will use the present value of these earnings in our expression of $W$. Thus each $e^{r(\tau - t_i)}$ will have present value $e^{-rt_i}$, leading us to the definition

$$W = (\Sigma_{i=1}^{N} Z^{(i)} e^{-rt_i}) I_{\{N \geq 1\}}. \tag{3.1}$$