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On the number of renewals in random time

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ABSTRACT

For the renewal counting process $M(t) = \min\{k : S_k > t\}$ and the independent of it non-negative random variable T, we investigate the asymptotic behaviour of $P(M(t) < T)$ and $P(M(t) \le K(t)x \mid M(t) < T)$ in cases when the interarrival times have an infinite mean. These quantities appear in a natural way when considering limiting behaviour of random time changed branching processes and shock models.

1. Introduction

Consider the discrete time stochastic process $\mathbf{Z} = \{Z(n), 0 \le n < T\}$, where the random variable T is positive integer valued. The variable T is called the life period of the process. In fact, the process evolves during the life period T and then dies out.

Consider also, an independent of $\mathbf{Z}$, sequence of nonnegative i.i.d. random variables $\mathbf{X} = \{X_1, X_2, \ldots\}$ with distribution function (d.f.) $F(x)$. Let $S_0 = 0$ and for $n \ge 1$, let $S_n = X_1 + X_2 + \cdots + X_n$. Define the renewal counting process

$$M(t) = \min\{k : S_k > t\}.$$

Using these two processes one can define a new process $\xi(t)$ as follows:

$$\xi(t) = Z(M(t)), \quad 0 \le M(t) < T,\ t \ge 0.$$

Clearly the subordinated process has a lifetime depending on T and $M(t)$.

In the present paper we are interested in the following problems.

Q1. What is the life period of the subordinated process? To this end we shall determine the probability of the event

$$\{\xi(t) \text{ is alive at time } t\} = \{M(t) < T\}.$$

Later we will determine the asymptotic behavior of $P(M(t) < T)$ as $t \to \infty$. Note that

$$P(M(t) < T) = E(\overline{F}_T(M(t))),$$

where $\overline{F}_T(t) = 1 - F_T(t) = P(T > t)$.

Q2. What happens with the subordinator during the life period? In this case we study probabilities of the form

$$P(M(t) \le x \mid M(t) < T).$$

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For an appropriate normalizing function $K(t)$, we will obtain the detailed asymptotic behaviour of

$$U_t(x) = P(M(t) \leq K(t)x \mid M(t) < T).$$

Clearly we have

$$1 - U_t(x) = \frac{P(M(t) > K(t)x, M(t) < T)}{P(M(t) < T)},$$

and we see that problem *Q2* is related to problem *Q1*.

In particular, one can consider $Z(n)$ as a Galton–Watson branching process in the subcritical or critical case (See e.g. Athreya and Ney, 1972). In these two cases the branching process evolves until reaching the state zero, i.e. it degenerates with probability one at random time T. The asymptotic of the probability for non-extinction and the limiting distributions are of primary interest for the branching processes. Let us remember that the limiting distributions are non degenerate only conditionally on the event $\{T > n\} = \{$*the process does not degenerate at time* $n\}$. In case of the random time changed branching process $Z(M(t))$, this event becomes $\{T > M(t)\}$ and the solutions of the questions *Q1* and *Q2* relate to the probability of non extinction and the limiting distribution of this process.

The paper is organized as follows. Section 2 contains some results from weighted renewal theory which are used later. In Section 3 we study the event $\{M(t) > T\}$ and use results from the weighted renewal theory to obtain the asymptotic behaviour of $P(M(t) > T)$ under various conditions on (the distribution of) $\mathbf{X}$ and T.

In Section 4 we study the asymptotic behaviour of $U_t(x) = P(M(t) \leq K(t)x \mid M(t) < T)$. We finish the paper with some concluding remarks.

In this paper we shall assume that $\mu = E(X) = \infty$. In the case where $\mu < \infty$, it is well-known that $M(t)/t \to 1/\mu$ and for this case there are many papers (cf. Alsmeyer (1992), Mallor and Omey (2006), Omey and Teugels (2002)) that give conditions under which

$$P(M(t) < T) = E(\overline{F}_T(M(t))) \sim \overline{F}_T(t/\mu).$$

We treat the more complicated case where $\mu = \infty$. As for the random variable T, we consider the two cases where either $E(T) < \infty$ or $E(T) = \infty$.

2. Weighted renewal theory

2.1. Notations

Let $X_1, X_2, \ldots$ denote i.i.d. positive r.v. with distribution function $F(x) = P(X_i \leq x)$. Let $a_k, k \geq 0$ denote a sequence of weights.

Now consider the weighted renewal function (wrf) $G(x)$ defined as follows:

$$G(x) = \sum_{k=0}^{\infty} a_k F^{*k}(x),$$

where, as usual, $F^{*k}(.)$ means the k-fold convolution of the d.f. $F(.)$.

The wrf $G(x)$ has a useful interpretation as an expectation. To see this, we define the function $A(x)$ as follows: $A(x) = a_0 + a_1 + \cdots + a_{[x]}, x \geq 0$. In this case we have $G(x) = E(A(M(x)))$, where $M(x)$ denotes the renewal counting measure defined above. Conversely, for any function $A(x)$ we have

$$E(A(M(x))) = \sum_{k=0}^{\infty} a_k F^{*k}(x),$$

where $a_0 = A(0)$ and $a_k = A(k) - A(k-1), k \geq 1$.

2.2. Asymptotic behaviour of $G(t)$

In the case where $\mu = \infty$, it makes sense to assume that the tail distribution $\overline{F}(x)$ is regularly varying. The motivation is in the following result. To formulate the result, define the integrated tail

$$m(x) = \int_0^x \overline{F}(t)\mathrm{d}t,$$

and let $K(x) = x/m(x)$. Note that if $\mu < \infty$, we have $K(x) \sim x/\mu$.

Proposition 1 (*Mallor and Omey, 2006, Theorems 6 and 9*).