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The uniform law for sojourn measures of random fields☆

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ABSTRACT

The uniform law for sojourn times of processes with cyclically exchangeable increments is extended to the case of random fields, with general parameter sets, that possess a suitable invariance property.

1. Introduction

Among the most interesting and important problems on the pathwise behaviour of random processes is the one on the distribution of the time spent by the trajectory of the process under a given (possibly random) line. The simplest question to be asked in that context is, of course, whether the trajectory of the process will cross that line at all. Historically, the first problem of that kind was perhaps the famous ballot theorem that effectively asserts that, for a simple symmetric random walk conditioned to be at point $-k < 0$ at time n, the probability that the trajectory will stay below zero on $(0, n]$ is k/n (for a historical background and the development of ballot theorems, see Takács, 1997). The result is an immediate consequence of a simple invariance property of the random walk and can readily be extended to random walks with cyclically exchangeable integer-valued jumps ≥ -1 (see e.g. in Takács (1997, Theorem 7.6.1)).

It turned out that there are more interesting distributional results for sojourn times that are also simple consequences of some invariance properties of the processes in question. One of them is the uniform law for the sojourn time of the negative half-axis in a simple symmetric random walk conditioned to be at zero at the terminal time, of which the continuous time analog is the uniformity of the distribution of the time spent below zero by the standard Brownian bridge on $[0, 1]$, the result going back to Lévy (1939) (that paper also contained the arc-sine law for the negative sojourn time of the Brownian motion on $[0, 1]$). These laws were later generalised to skew Brownian motion and skew Bessel processes, with positive sojourn times that follow a generalised arc-sine law (Lamperti, 1958; Barlow et al., 1989); see also Watanabe, 1995 for a characterisation result.

In the mid-1990s, the uniform laws for sojourn times were extended to suitable Lévy bridges in Fitzsimmons and Getoor (1995), and necessary and sufficient conditions for such laws were provided for bridge processes with exchangeable

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increments in Knight (1996). It appears that Kallenberg (1999) was the first paper to note that the argument from Knight (1996) also extends (with conditions) to any measurable bridge process with cyclically exchangeable increments (or, equivalently, any measurable stationary periodic processes on $\mathbb{R}_+$).

The nice invariance property exploited in Knight (1996) allows one to vastly simplify arguments proving some seemingly complex results (the reader may wish to compare the proofs in Knight, 1996 with those in Fitzsimmons and Getoor, 1995). This invariance property is discussed in Chaumont (2000) and Chaumont et al. (2001), and is due to the cyclic exchangeability of increments which is defined as follows.

Let $X = \{X(t) : t \in [0, 1]\}$ be a measurable real-valued stochastic process. We say that X has cyclically exchangeable increments if, for any $u \in [0, 1]$, the process

$$X_u(t) := \begin{cases} X(u+t) - X(u) + X(0) & \text{for } 0 \le t < 1-u, \\ X(1) - X(u) + X(u+t-1) & \text{for } 1-u \le t \le 1, \end{cases}$$

has the same distribution as X. The simplest examples of such objects are Lévy processes and bridges, and the uniform empirical processes.

Introduce the "sojourn function"

$$F(X, x) = \lambda\big(t \in [0, 1] : X(t) \le x\big), \quad x \in \mathbb{R},$$

where λ is the Lebesgue measure on $\mathbb{R}$. The value $F(X, x)$ is the (random) duration of time the process X spent below or at the level x. Then the following result on the uniformity of the negative sojourn time holds true (see e.g. Kallenberg, 1999 or Chaumont et al., 2001).

Theorem 1. *If X has cyclically exchangeable increments, $X(0) = X(1) = 0$ and $F(X, \cdot)$ is continuous a.s., then $F(X, 0) \sim U(0, 1)$.*

In what follows, we will be interested in situations where $X(0) = X(1)$, in which case the definition of X_u simplifies to

$$X_u(t) := X(t + u (\text{mod } 1)) - X(u) + X(0), \quad t \in [0, 1]. \tag{1}$$

In that case, it is more convenient to view the process X as given on the unit circle $\mathbb{S}^1$ using, say, the natural complex number parametrisation

$$\widetilde{X}(e^{2\pi i t}) := X(t), \quad t \in [0, 1).$$

Moreover, assuming integrability of X, observe that the cyclic exchangeability of increments now translates into the invariance, with respect to rotations of the parametric set $\mathbb{S}^1$, of the distribution of the process $\{\widetilde{X}^0(z) : z \in \mathbb{S}^1\}$ given by

$$\widetilde{X}^0(e^{2\pi i t}) := X(t) - \int_0^1 X(s)\, ds, \quad t \in [0, 1).$$

Indeed, note that, in view of (1) and the cyclic exchangeability property that $X_u(\cdot) \stackrel{d}{=} X(\cdot)$, we have, for any $u \in [0, 1)$, the relations

$$\begin{aligned} \widetilde{X}^0\big(e^{2\pi i(\cdot + u\ (\text{mod} 1))}\big) &= X(\cdot + u(\text{mod } 1)) - \int_0^1 X(s + u(\text{mod } 1))\, ds \\ &= X_u(\cdot) - \int_0^1 X_u(s)\, ds \\ &\stackrel{d}{=} X(\cdot) - \int_0^1 X(s)\, ds = \widetilde{X}^0\big(e^{2\pi i \cdot}\big). \end{aligned}$$

Now the point $t = 0$ ceases to be special since, for any fixed $a \in \mathbb{S}^1$, the distribution of the "time" spent by $\widetilde{X}^0$ below the level $\widetilde{X}^0(0)$ coincides with that of the time spent by $\widetilde{X}^0$ below the level $\widetilde{X}^0(a)$, and in view of the result of Theorem 1, that distribution is uniform.

The objective of the present note is to demonstrate how the above result can, in a natural (and rather elementary) way, be extended to the random fields setting where the parametric set T of the field $\{X(t) : t \in T\}$ is endowed with a finite measure μ. We show that, for a fixed $a \in T$, provided that a certain invariance property is satisfied, the μ-measure of the subset of T on which the values of X do not exceed that of $X(a)$, is also uniformly distributed. In a sense, this result is a "randomised" version of the well-known fact that, for a random variable ξ with continuous distribution function H, one has $H(\xi) \sim U(0, 1)$.

2. The main result

Let $(T, \mathscr{T}, \mu)$ be a measure space with a finite measure and $\{X(t) : t \in T\}$ a real-valued measurable random field on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with parameter set T. Without loss of generality, we will assume that $\mu(T) = 1$. Further, let

$$F_\mu(X, x) := \mu\big(t \in T : X(t) \le x\big), \quad x \in \mathbb{R},$$

be the μ-measure of the parameter values t for which $X(t)$ was at or below the level x.