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How rich is the class of processes which are infinitely divisible with respect to time?

Khalifa Es-sebaiy, Youssef Ouknine*

Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, 2390 Marrakesh, Morocco

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Abstract

We give a link between stochastic processes which are infinitely divisible with respect to time (IDT) and Lévy processes. We investigate the connection between the selfsimilarity and the strict stability for IDT processes. We also consider a subordination of a Lévy process by an increasing IDT process. We introduce a notion of multiparameter IDT stochastic processes, extending the one studied by Mansuy [2005. On processes which are infinitely divisible with respect to time. arXiv:math/0504408.]. The main example of this kind of processes is the Lévy sheet.

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1. Introduction

An $\mathbb{R}^d$-valued stochastic process $X = (X_t, t \geqslant 0)$ is said to be infinitely divisible with respect to time (IDT) if, for every $n \in \mathbb{N}$, we have

$$(X_{nt}, t \geqslant 0) \stackrel{d}{=} (X_t^{(1)} + \cdots + X_t^{(n)}, t \geqslant 0),$$

where $(X^{(1)}, t \geqslant 0), \ldots, (X^{(n)}, t \geqslant 0)$ are independent copies of X and $\stackrel{d}{=}$ denotes equality in all finite-dimensional distributions. The notion of IDT processes has been introduced by Mansuy (2005) as a generalization of Lévy processes. Various properties of IDT processes have been already investigated in Mansuy (2005), related for instance to their temporal self-decomposability and the characterization of IDT Gaussian processes. Regarded as a contribution to this expending topic, it is the purpose of this paper to extend some results on Lévy processes studied in Barndorff-Nielsen et al. (2006), Embrechts and Meajima (2002) and Meajima and Sato (1999) to the case of IDT processes. In particular, we shall prove that, IDT processes are more tractable than Lévy processes, since they could be obtained by combining the selfsimilarity and strict stability. A such result is not true in general for Lévy processes. Moreover, we will prove that a necessary and sufficient condition for

*Corresponding author.
E-mail addresses: k.essebaiy@ucam.ac.ma (K. Es-sebaiy), ouknine@ucam.ac.ma (Y. Ouknine).

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an IDT process to be a Lévy process is the hypothesis of independence increments. While, this condition can be circumvented when dealing with IDT processes. So it turns out that the class of IDT processes can be very rich.

The paper is organized as follows. Section 2 contains some preliminaries on stable processes and selfsimilar processes. Section 3 establishes, for IDT processes, the connection between the selfsimilar (semi-selfsimilar, resp.) processes and the strictly stable (strictly semi-stable, resp.) processes. Namely, strictly stable (strictly semi-stable, resp.) IDT process is a simple example of selfsimilar (semi-selfsimilar, resp.) process. As a byproduct, we consider the so-called Lamperti transformation for strictly semi-stable IDT processes to give a generalized semi-stable Ornstein–Uhlenbeck process (see Definition 3.4).

Time-changed Lévy processes where the chronometers are more general than subordinators arise now in many fields of application, see for instance Barndorff-Nielsen et al. (2006) and the references therein. We shall prove (Theorem 3.6) the inheritance of IDT under time change when base processes are Lévy processes.

In Section 4 we shall introduce a notion of multiparameter IDT processes and we give several examples of this kind of processes, one of them is the Lévy sheet. Contrary to the one-parameter case, we will prove that multiparameter Lévy processes are not IDT in our sense. As in the one-parameter case (Mansuy, 2005), we characterize the multiparameter IDT Gaussian processes. Moreover, we define multiparameter temporal selfdecomposable processes similar to those introduced by Barndorff-Nielsen et al. (2006) and we prove that multiparameter IDT processes are temporal selfdecomposable.

2. Preliminaries

In this Section we recall some definitions that we will use in the sequel. For more details the reader is referred to Sato (1999).

An $\mathbb{R}^d$-valued random variable X is called degenerate if it is a constant almost surely. An $\mathbb{R}^d$-valued process $(X_t, t\geqslant 0)$ is called trivial if X_t is degenerate for every t.

Let $0<\alpha\leqslant 2$. An infinitely divisible probability measure μ on $\mathbb{R}^d$ is called α-stable if, for any $a>0$, there is $\gamma_a \in \mathbb{R}^d$ such that

$$\hat{\mu}(\theta)^a := \left(\int_{\mathbb{R}^d} e^{i\langle\theta,z\rangle}\mu(dz)\right)^a = \hat{\mu}(a^{1/\alpha}\theta)e^{i\langle\theta,\gamma_a\rangle}, \quad \forall\theta\in\mathbb{R}^d. \tag{1}$$

It is called strictly α-stable if, for any $a>0$,

$$\hat{\mu}(\theta)^a = \hat{\mu}(a^{1/\alpha}\theta), \quad \forall\, \theta\in\mathbb{R}^d. \tag{2}$$

It is called α-semi-stable if, for some $a>0$ with $a\neq 1$, there is $\gamma_a\in\mathbb{R}^d$ satisfying (1). It is called strictly α-semi-stable if, there is some $a>0$ with $a\neq 1$ satisfying (2).

Let $(X_t, t\geqslant 0)$ be a Lévy (IDT, resp.) process on $\mathbb{R}^d$. It is called a α-stable, strictly α-stable, semi-α-stable, or strictly α-semi-stable Lévy (IDT, resp.) process if every finite-dimensional distribution of X is, respectively, α-stable, strictly α-stable, semi-α-stable, or strictly α-semi-stable.

Let $H>0$. A stochastic process $(X_t, t\geqslant 0)$ on $\mathbb{R}^d$ is called H-selfsimilar if, for any $a>0$,

$$(X_{at}, t\geqslant 0) \stackrel{d}{=} (a^H X_t, t\geqslant 0). \tag{3}$$

It is called wide-sense H-selfsimilar if, for any $a>0$, there is a function $c(t)$ from $\mathbb{R}^+$ to $\mathbb{R}^d$ such that

$$(X_{at}, t\geqslant 0) \stackrel{d}{=} (a^H X_t + c(t), t\geqslant 0). \tag{4}$$

It is called H-semi-selfsimilar if, there is some $a>0$ with $a\neq 1$ satisfying (3) and it is called wide-sense H-semi-selfsimilar if, for some $a>0$ with $a\neq 1$, there is a function $c(t)$ satisfying (4).

3. Stable IDT processes

The goal of this section is to generalize some properties of Lévy process to the case of IDT process. We first establish a link between IDT process and Lévy process than between IDT process and selfsimilar process.