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A necessary and sufficient condition for probability measures dominated by g-expectation☆

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ABSTRACT

We prove that if the generator g of a g-expectation $\mathcal{E}_g$ is independent of y, then a probability measure Q is dominated by $\mathcal{E}_g$ if and only if Q can be generated by the Girsanov transformation via a controlled process; furthermore, we prove that $\mathcal{E}_g$ equals the supremum of its dominated mathematical expectations if and only if g is sublinear with respect to z.

1. Introduction and preliminaries

1.1. Introduction

By Pardoux and Peng (1990) we know that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE for short) of the type

$$y_t = \xi + \int_t^T g\left(s, y_s, z_s\right) \mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad 0 \le t \le T, \tag{1}$$

provided the function g is Lipschitz in both variables y and z, and ξ and $(g\left(t, 0, 0\right))_{t\in[0,T]}$ are square integrable. g is said to be the generator of the BSDE (1). We denote the unique adapted and square integrable solution of the BSDE (1) by $\left(Y_t\left(g, T, \xi\right), Z_t\left(g, T, \xi\right)\right)_{t\in[0,T]}$. When g also satisfies $g(\cdot, y, 0) \equiv 0$ for any y, then $Y_0\left(g, T, \xi\right)$, denoted by $\mathcal{E}_g[\xi]$, is called the g-expectation of ξ; $Y_t\left(g, T, \xi\right)$, denoted by $\mathcal{E}_g[\xi|\mathcal{F}_t]$, is called the conditional g-expectation of ξ (*see* Peng (1997)).

g-expectation is a kind of nonlinear expectation, which can be considered as a nonlinear extension of the well-known Girsanov transformations. The original motivation for studying g-expectation comes from the theory of expected utility. Since the notion of g-expectation was introduced, many properties of g-expectation have been studied in Peng (1997, 2004), Chen (1998), Briand et al. (2000), Chen and Epstein (2002), Coquet et al. (2002), Chen et al. (2003a,b, 2005), Jiang (2005, 2006), Hu (2005) and Rosazza Gianin (2006). Chen and Epstein (2002) gave an application of g-expectation to recursive utility. Rosazza Gianin (2006) introduced some examples of dynamic risk measures via g-expectations.

Let g be independent of y and $g(t, 0) \equiv 0$. We define two sets of probability measures on the measurable space $(\Omega, \mathcal{F}_T)$:

$$\mathcal{S}_1^g := \left\{Q : E_Q[\xi] \le \mathcal{E}_g[\xi], \forall \xi \in L^2\left(\Omega, \mathcal{F}_T, P\right)\right\}, \tag{2}$$

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$$\mathcal{S}_2^g := \left\{ Q^\theta : \theta \in \Theta^g, \text{ and } \left. \frac{dQ^\theta}{dP} \right|_{\mathcal{F}_t} = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right), t \in [0, T] \right\}, \tag{3}$$

where

$$\Theta^g := \left\{ (\theta_t)_{t\in[0,T]} : \begin{array}{l} \theta \text{ is } \mathbf{R}^d\text{-valued, progressively measurable and} \\ dP \times dt \text{ -a.s., } \forall z \in \mathbf{R}^d,\ \theta_t \cdot z \leq g(t, z) \end{array} \right\}. \tag{4}$$

In this paper we want to investigate whether $\mathcal{S}_1^g = \mathcal{S}_2^g$ and what conditions should be given to g such that

$$\mathcal{E}_g[\xi] = \sup_{Q\in\mathcal{S}_1^g} E_Q[\xi], \quad \forall \xi \in L^2(\Omega, \mathcal{F}_T, P). \tag{5}$$

We will prove that $\mathcal{S}_1^g = \mathcal{S}_2^g$ indeed, and we will also prove that equality (5) holds if and only if g is sublinear with respect to z.

The remainder of this paper is organized as follows. In Section 1.2, we introduce some assumptions, definitions and lemmas. In Section 2, we state and prove our main results.

1.2. Preliminaries

Let $T > 0$ be a given real number; let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $(B_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion on this space such that $B_0 = 0$; let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by the Brownian motion $(B_t)_{t\geq 0}$ and augmented by the set of all P-null subsets. For any positive integer n and $z \in \mathbf{R}^n$, $|z|$ denotes its Euclidean norm.

We define the following usual spaces of processes:

$\mathcal{S}_{\mathcal{F}}^2(0, T; \mathbf{R}) := \{\psi$ continuous and progressively measurable; $\mathbf{E}\left[\sup_{0\leq t\leq T} |\psi_t|^2\right] < \infty\}$;

$\mathcal{H}_{\mathcal{F}}^2(0, T; \mathbf{R}^n) := \{\psi$ progressively measurable; $\|\psi\|_2^2 = \mathbf{E}\left[\int_0^T |\psi_t|^2 dt\right] < \infty\}$.

A generator g of a BSDE is a function

$$g : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \longrightarrow \mathbf{R}$$

such that the process $(g(t, y, z))_{t\in[0,T]}$ is progressively measurable for each pair (y, z) in $\mathbf{R} \times \mathbf{R}^d$, and furthermore, g satisfies the following assumptions (A1) and (A2):

(A1) There exists a constant $K \geq 0$, such that $dP \times dt$ -a.s.,

$$\forall y_1, y_2 \in \mathbf{R}, z_1, z_2 \in \mathbf{R}^d, \quad |g(t, y_1, z_1) - g(t, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|).$$

(A2) The process $(g(t, 0, 0))_{t\in[0,T]} \in \mathcal{H}_{\mathcal{F}}^2(0, T; \mathbf{R})$.

(A3) $dP \times dt$ -a.s., $\forall y \in \mathbf{R}$, $g(t, y, 0) \equiv 0$.

Let g satisfy (A1) and (A2). Then for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, there exists a unique pair of processes in $\mathcal{S}_{\mathcal{F}}^2(0, T; \mathbf{R}) \times \mathcal{H}_{\mathcal{F}}^2(0, T; \mathbf{R}^d)$, denoted by $(Y_t(g, T, \xi), Z_t(g, T, \xi))_{t\in[0,T]}$, solving the BSDE (1) (see Pardoux and Peng (1990)).

For the convenience of the reader, we recall the notion of g-expectation and conditional g-expectation defined in Peng (1997). We also list some properties of g-expectations. In the following Definitions 1.1 and 1.2, we always assume that g satisfies (A1) and (A3).

Definition 1.1. The g-expectation $\mathcal{E}_g[\cdot] : L^2(\Omega, \mathcal{F}_T, P) \longmapsto \mathbf{R}$ is defined by

$$\mathcal{E}_g[\xi] = Y_0(g, T, \xi).$$

Definition 1.2. The conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = Y_t(g, T, \xi).$$

Lemma 1.1 comes from Peng (1997), where g is assumed to satisfy (A1) and (A3).

Lemma 1.1. *$\mathcal{E}_g[\xi|\mathcal{F}_t]$ is the unique random variable η in $L^2(\Omega, \mathcal{F}_t, P)$ such that*

$$\mathcal{E}_g[\xi 1_A] = \mathcal{E}_g[\eta 1_A], \quad \textit{for all } A \in \mathcal{F}_t.$$

The following Lemma 1.2 is a special case of Lemma 1.4 in Jiang (2006).

Lemma 1.2. *Let g satisfy* (A1) *and* (A3)*; let g be independent of y, that is, g is defined on $\Omega \times [0, T] \times \mathbf{R}^d$. Let $1 \leq p < 2$. Then for any $z \in \mathbf{R}^d$,*

$$g(t, z) = L^p - \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \mathcal{E}_g[z \cdot (B_{t+\varepsilon} - B_t) | \mathcal{F}_t], \quad \textit{a.e. } t \in [0, T[.$$