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A note on absorption probabilities in one-dimensional random walk via complex-valued martingales

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Abstract

Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables taking values in a finite set of integers, and let $S_n = S_{n-1} + X_n$ for $n \geqslant 1$ and $S_0 = 0$ be a random walk on $\mathbb{Z}$, the set of integers. By using the zeros, together with their multiplicities, of the rational function $f(x) = \mathbb{E}(x^X) - 1, x \in \mathbb{C}$, we characterize the space U of all complex-valued martingales of the form $\{g(S_n), n \geqslant 0\}$ for some function $g : \mathbb{Z} \to \mathbb{C}$. As an application we calculate the absorption probabilities of the random walk $\{S_n, n \geqslant 0\}$ by applying the optional stopping theorem simultaneously to a basis of the martingale space U. The advantage of our method over the classical approach via the Markov chain techniques (cf. Kemeny and Snell [1960. Finite Markov Chains. Van Nostrand, Princeton, NJ.]) is in the size of the matrix that is needed to be inverted. It is much smaller by our method. Some examples are presented.

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Keywords: Random walks; Martingales; Optional stopping theorem; Absorption probabilities

1. Introduction and main results

We deal here with a random walk $\{S_n, n \geqslant 0\}$ on $\mathbb{Z}$. Specifically,

$$S_n = S_{n-1} + X_n \quad \text{for } n \geqslant 1 \quad \text{and} \quad S_0 = 0,$$

where $\{X, X_n, n \geqslant 1\}$ are i.i.d. random variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in a finite subset of $\mathbb{Z}$. We will consider three cases:

Case 1: Two-sided random walk:

$$p_i = \mathbb{P}(X = i), \quad i = 0, 1, \ldots, a,$$
$$q_i = \mathbb{P}(X = -i), \quad i = 1, 2, \ldots, b,$$

where $p_a > 0, q_b > 0$ and $\sum_{i=0}^{a} p_i + \sum_{i=1}^{b} q_i = 1$.

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Case 2: Right-sided random walk:

$$p_i = \mathbb{P}(X = i), \quad i = 0, 1, \ldots, a,$$
$$\sum_{i=0}^{a} p_i = 1 \quad \text{and} \quad p_a > 0.$$

Case 3: Left-sided random walk:

$$q_i = \mathbb{P}(X = -i), \quad i = 0, 1, \ldots, b,$$
$$\sum_{i=0}^{b} q_i = 1 \quad \text{and} \quad q_b > 0.$$

In all the cases, without loss of generality for our purpose, we assume that the integers in $I \equiv \{k \in \mathbb{Z} : \mathbb{P}(X = k) > 0\}$ do not have a common divisor larger than 1. (If not, then $S_n \in r\mathbb{Z}$ where $r > 1$ is the largest common divisor of I and we will replace X by X/r, etc.)

Let $\mathbb{A}$ be the collection of all integers that $\{S_n, n \geqslant 0\}$ will ever visit with positive probability, i.e.,

$$\mathbb{A} = \left\{ m \in \mathbb{Z} : \sum_{n=0}^{\infty} \mathbb{P}(S_n = m) > 0 \right\}.$$

It follows from elementary results in number theory that in Cases 1, 2 and 3, respectively,

$$\mathbb{A} = \mathbb{Z},$$
$$\mathbb{Z}^+ \supseteq \mathbb{A} \supseteq \{m \geqslant m_0\} \quad \text{for some } m_0 \geqslant 0 \quad \text{and}$$
$$\mathbb{Z}^- \supseteq \mathbb{A} \supseteq \{m \leqslant m_1\} \quad \text{for some } m_1 \leqslant 0,$$

where $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^- = \{0, -1, -2, \ldots\}$.

In order to describe our results, let

$$f(x) = \mathbb{E}(x^X) - 1, \quad x \in \mathbb{C} \backslash \{0\},$$

where $\mathbb{C}$ denotes the field of complex numbers. Corresponding to the three cases, we have

$$f(x) = \sum_{i=0}^{a} p_i x^i - 1 + \sum_{i=1}^{b} q_i x^{-i},$$
$$f(x) = \sum_{i=0}^{a} p_i x^i - 1,$$
$$f(x) = \sum_{i=0}^{b} q_i x^{-i} - 1.$$

Let $\{x_j$ with multiplicity $k_j, j \in J\}$ denote the roots, over $\mathbb{C}$, of $f(x) = 0$. It follows that in the three cases, $\sum_{j \in J} k_j = a + b, a, b$, respectively.

Our first result characterizes U the space of all functions $g : \mathbb{A} \to \mathbb{C}$ such that $\{g(S_n), n \geqslant 0\}$ is a martingale with respect to the filtration $\{\mathscr{F}_n, n \geqslant 0\}$, where $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $\mathscr{F}_n \equiv \sigma(X_1, \ldots, X_n)$ for all $n \geqslant 1$. When there is no confusion, we will also call U the space of all martingales of the form $\{g(S_n), n \geqslant 0\}$.

Proposition 1.1. *In the three cases, U is a linear space over $\mathbb{C}$ with dimension $a + b$, a and b, respectively. A basis for U is given by*

$$\{g^{j,l}(\cdot) : l = 0, 1, \ldots, k_j - 1; j \in J\}, \tag{1.1}$$

where $g^{j,l}(\cdot) : \mathbb{A} \to \mathbb{C}$ is the function defined by $g^{j,l}(m) = (x_j)^m m^l, \forall m \in \mathbb{A}$. Here and in the sequel, we use the convention $0^0 = 1$.