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The rank of a normally distributed matrix and positive definiteness of a noncentral Wishart distributed matrix

A.G.M. Steerneman[✠], Frederieke van Perlo-ten Kleij[*,1]

Department of Econometrics, University of Groningen, Landleven 5, P.O. Box 800, 9700 AV Groningen, The Netherlands

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Abstract

If $X \sim \mathcal{N}_{n\times k}(M, I_n \otimes \Sigma)$, then $S = X'X$ has the noncentral Wishart distribution $W'_k(n, \Sigma; \Lambda)$, where $\Lambda = M'M$. Here Σ is allowed to be singular. It is well known that if $\Lambda = 0$, then S has a (central) Wishart distribution and S is positive definite with probability 1 if and only if $n \geqslant k$ and Σ is positive definite. We show that if S has a noncentral Wishart distribution, then S is positive definite with probability 1 if and only if $n \geqslant k$ and $\Sigma + \Lambda$ is positive definite. This is a consequence of the main result that $\operatorname{rank} X = \min(n, \operatorname{rank}(\Sigma + \Lambda))$ with probability 1.

Keywords: Multivariate normal distribution; Singular covariance matrix; Rank; Noncentral Wishart; Positive definite

1. Introduction

It is well known that if $S \sim W_k(n, \Sigma)$, then S is positive definite with probability 1 if and only if $n \geqslant k$ and Σ is positive definite, see, for example, Muirhead (1982), Theorem 3.1.4 and Eaton (1983), Proposition 8.2. The proof of this result goes back to Dijkstra (1970). In case S follows a noncentral Wishart distribution, S is also positive definite if $n \geqslant k$, see Muirhead (1982), page 442 and Eaton (1983) states in page 317 that S is positive definite with probability 1 if and only if $n \geqslant k$ and Σ is positive definite. Eaton and Perlman (1973) study the non-singularity of (generalized) sample covariance matrices in great detail. To the best of our knowledge, the literature thus gives the following result: if Σ is positive definite, then S is positive definite with probability 1 if and only if $n \geqslant k$. We focus on noncentral singular Wishart distributions and drop the condition that Σ is positive definite. For the (central) singular Wishart distribution in particular, reference can be made to Uhlig (1994) and Srivastava (2003).

In Section 2 we start with some basic notation. In Section 3 we prove a theorem on the rank of $X \sim \mathcal{N}_{n\times k}(M, I_n \otimes \Sigma)$, which is the basis for our main result on the positive definiteness of the noncentral Wishart matrix S. We present and prove this result in Section 4.

*Corresponding author.
E-mail address: fvanperlo@hotmail.com (F. van Perlo-ten Kleij).
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2. Notation

If the random $k \times 1$ vector $X \sim \mathcal{N}_k(\mu, \Sigma)$, where $\mu \in \mathbb{R}^k$ and Σ is a positive semi-definite symmetric $k \times k$ matrix, then we say that X follows a multivariate normal distribution with mean μ and covariance matrix Σ. We allow Σ to be singular, which includes the possibility that X lies in an affine subspace with probability 1. In the above notation, the distribution of X is degenerated if $k < \operatorname{rank} \Sigma$.

When studying Wishart distributions, notational comfort is provided by using random matrices whose elements are multivariate normally distributed. The random $n \times k$ matrix X follows the $\mathcal{N}_{n \times k}(M, \Omega)$ distribution if

$$\operatorname{vec} X' \sim \mathcal{N}_{nk}(\operatorname{vec} M', \Omega),$$

where $M = \mathrm{E}X$ is an $n \times k$ matrix and Ω is the $nk \times nk$ covariance matrix of $\operatorname{vec} X'$. If the rows $X_1, \ldots, X_n$ of the random $n \times k$ matrix $X = (X_1, \ldots, X_n)'$ are independently distributed as $\mathcal{N}_k(\mu_i, \Sigma)$, then

$$X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma),$$

where $M = (\mu_1, \ldots, \mu_n)'$.

Definition 1. Let $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$, then $S = X'X$ has the noncentral Wishart distribution of dimension k, degrees of freedom n, covariance matrix Σ and noncentrality matrix $\Lambda = M'M$. This distribution is denoted by $W_k'(n, \Sigma; \Lambda)$. If $M = 0$, then S has the Wishart distribution, denoted by $W_k(n, \Sigma)$.

3. The rank of a normally distributed matrix

Frequently we assume that we have n independent k-variate normal observations assembled in a matrix X, denoted as $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$, where Σ is positive semi-definite. Usually, the number of observations n will be larger than k, but we will not adopt this assumption, see also Srivastava (2003). If the rank of Σ is known, what can be said about the rank of X? The answer is given in the following theorem.

Theorem 1. *If* $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$, *then*

$$\operatorname{rank} X = \min(n, \operatorname{rank}(\Sigma + M'M))$$

with probability 1.

Before we prove Theorem 1 we want to emphasize the importance of this theorem. In the one-dimensional case, where $X \sim \mathcal{N}(\mu, \sigma^2)$, we know that $X \neq 0$ with probability 1 either if $\mu \neq 0$, or if $\sigma^2 > 0$. Or, stated otherwise, $X \neq 0$ with probability 1 if and only if $\sigma^2 + \mu^2 > 0$, which corresponds with Theorem 1. Theorem 1 shows that, even if $\operatorname{rank} \Sigma < k$, the rank of X can still be equal to k, as long as the matrix of expectations M is such that $\operatorname{rank}(\Sigma + M'M) = k$, provided $n \geqslant k$.

The proof of this theorem is divided into a couple of lemmas. We use the notation of Section 2. In Lemma 1 we first consider the case $\operatorname{rank} \Sigma = k$. The proof of this lemma can be found in Srivastava and Khatri (1979, p. 73).

Lemma 1. *If* $X \sim \mathcal{N}_{n \times k}(M, I_n \otimes \Sigma)$, *where* $\operatorname{rank} \Sigma = k$, *then* $\operatorname{rank} X = \min(n, k)$ *with probability* 1.

Lemma 2. *Let* $Y \sim \mathcal{N}_{n \times r}(M_1, I_n \otimes \Omega)$ *with* $\operatorname{rank} \Omega = r$, *let* M_2 *be an* $n \times s$ *matrix of rank* s, *and let* $X = (Y, M_2)$. *Then* $\operatorname{rank} X = \min(n, r + s)$ *with probability* 1.

Proof. Note that the case $s = 0$ is covered by Lemma 1. So, we may assume $s > 0$. Obviously $s \leqslant n$.

(i) Assume $n \leqslant r$. According to Lemma 1 we have

$$1 = \mathrm{P}\{\operatorname{rank} Y = n\} = \mathrm{P}\{\operatorname{rank}(Y, M_2) = n\} = \mathrm{P}\{\operatorname{rank} X = n\}.$$

(ii) Assume $r < n \leqslant r + s$. Let $M_2 = (m_{r+1}, \ldots, m_{r+s})$. Define $Z = XA$, where the $(r+s) \times (r+s)$ matrix

$$A = \begin{pmatrix} \Omega^{-1/2} & 0 \\ 0 & I_s \end{pmatrix}$$