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Lattice polynomials of random variables

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Abstract

In many statistics and reliability theory models the object of interest is a random variable obtained from others by minimum and maximum operations. As a generalization, a random variable Y defined as a lattice polynomial of random arguments was introduced in Marichal [2006. Cumulative distribution function and moments of lattice polynomials. Statist. Probab. Lett. 76(12), 1273–1279] and studied in case of independent identically distributed arguments. Here, the cumulative distribution function of Y (in particular, order statistic) is studied for generally dependent arguments and special cases. A relation (presented in [Marichal, 2006. Cumulative distribution function and moments of lattice polynomials. Statist. Probab. Lett. 76(12), 1273–1279]) between Y and order statistics is proved to hold if and only if the arguments possess "cardinality symmetry".

Keywords: Lattice polynomial; Order statistic; Probability generating function

1. Introduction

Random variables constructed from random arguments using minimum and maximum operations appear naturally in such applications as, for example, the lifetime of a control system or an order statistic of a random sample. More generally, in Marichal (2006), lattice polynomials were introduced as a way to aggregate random arguments. Starting from the seminal work of Birkhoff (1967), lattice polynomials have been given much attention in the literature. In particular, order statistics can be presented as (symmetric) lattice polynomials (see Ovchinnikov, 1998). That connection opens a way to explore order statistics under assumptions more general than so far (see, e.g., David and Nagaraja, 2003, for an extensive review of current results for order statistics—all under the assumption that the variates are either independent or exchangeable).

In Marichal (2006) the cumulative distribution function (c.d.f.) was obtained of a random variable Y defined by a lattice polynomial p of independent random arguments (c.d.f. of the kth order statistic, in particular). In the case of independent identically distributed (i.i.d.) arguments, the c.d.f. of Y was expressed as a linear combination of c.d.f.s of the order statistics with constant coefficients determined by the polynomial p.

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Both in reliability theory and statistical applications, the assumption of independent arguments is, in many situations, unrealistic. In this paper, a new approach is developed in order to obtain results similar to Marichal (2006) for generally dependent arguments.

The starting point is the definition of a lattice polynomial (as given in Marichal, 2006):

Given a finite collection of variables $x_1, \ldots, x_n$, that are elements of a general lattice, a lattice polynomial in those variables is defined as follows:

Definition 1. (1) The variables themselves are lattice polynomials;
(2) if p and q are lattice polynomials, then so are also $p \wedge q$ and $p \vee q$;
(3) every lattice polynomial is formed by finitely many applications of rules (1) and (2).

Treating R as a lattice where $\wedge$ and $\vee$ present as min and max, a random variable Y can be obtained as a lattice polynomial p of random variables $X_1, \ldots, X_n$:

$$Y = p(X_1, \ldots, X_n).$$

The approach developed here is suggested by the following interpretation of Y:

Consider a control system consisting of units with random "lifetimes" X_i (possibly, same for different units). For units connected in series, the lifetime of that segment is the minimum of their lifetimes, for parallel connections the lifetime is the maximum. Connecting the units into a system according to the lattice polynomial p, Y can be interpreted as the lifetime of the entire system. In those terms, $Y > y$ is the event that by the time y the system is still on, which in turn, is determined by the set of units that are on at the time y, that is, by the indicators of events $X_i > y$.

Now, the key fact here is that to analyze the system the joint probability distribution is needed of those indicator variables synchronized at the same time y, not of the entire set $X_1, \ldots, X_n$. Since the indicator variables are discrete (in fact, $\{0, 1\}$ valued), probability generating functions (p.g.f.) are used to represent the probability distributions.

In Section 2, the main results are presented on the c.d.f. of Y (in particular, order statistics) when the arguments are generally dependent. For the case of independent arguments the general formulas yield the same results as in Marichal (2006).

In Section 3, we specify the main results to the case where the arguments possess a "cardinality symmetry" property which, we prove, is necessary and sufficient for the relation between the c.d.f.s of and order statistics that was presented in Marichal (2006) for i.i.d. arguments. (As an example, some results of Marichal (2006) for i.i.d. arguments are reproduced.) Compared to the "exchangeable arguments" condition, "cardinality symmetry" applies to a wider class of cases where only the synchronous indicator variables need to be exchangeable.

2. The cumulative distribution function in the general case

In view of the above analogy with the lifetime of a unit, for a real-valued random variable X we introduce a supplementary indicator variable

$$\chi = \mathrm{Ind}(X > x).$$

For a set of variables $X_1, \ldots, X_n$, we denote the set of indices $[n] = \{1, \ldots, n\}$, and consider a vector of (synchronous) indicator variables

$$\vec{\chi}(x) = (\chi_1(x), \ldots, \chi_n(x)),$$

where

$$\chi_i(x) = \mathrm{Ind}(X_i > x), \quad i \in [n].$$

Let $Y = p(X_1, \ldots, X_n)$ and denote $\chi_Y(y) = \mathrm{Ind}(Y > y)$. The analogy with a control system leads to the following theorem.

Theorem 1.

$$\chi_Y(y) = p(\chi_1(y), \ldots, \chi_n(y)). \tag{1}$$