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A CLT for renewal processes with a finite set of interarrival distributions

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ABSTRACT

We prove a central limit theorem for a renewal process based on a sequence of independent non-negative interarrival times whose distributions are taken from a finite set. The result extends the classical central limit theorem obtained by Takács (1956).

1. Introduction

We consider a renewal process $N(t),\ t \geq 0$, based on a sequence $X_n,\ n \in \mathbb{N}$, of independent non-negative random variables, where the interarrival times $X_n, n \in \mathbb{N}$, have a finite number of possible distributions. More precisely, the set $\mathbb{N} = \{1, 2, \ldots\}$ is partitioned into p infinite subsets $\mathbb{N}_1, \ldots, \mathbb{N}_p$, and assume that there exists a finite collection of random variables $Y_1, \ldots, Y_p$ such that the distribution of X_n coincides with that of Y_i whenever $n \in \mathbb{N}_i$, $1 \leq i \leq p$. In this case, following Durrett et al. (1991) and Kesten and Lawler (1992), the sequence $S_0 = 0, S_n = X_1 + \cdots + X_n, n \in \mathbb{N}$, is called a finitely inhomogeneous random walk. A summary of results on finitely inhomogeneous random walks can be found in Spătaru (2010). Some classical results, of interest for renewal theory, can be generalized to the case $p \geq 2$. (For these generalizations, the assumption $X_n \geq 0$ is dropped.) Actually, if $E|X_n|^s < \infty, n \in \mathbb{N}$, then $S_n/n^{1/s} \xrightarrow{a.s.} 0$ for $0 < s < 1$, while $(S_n - ES_n)/n^{1/s} \xrightarrow{a.s.} 0$ for $1 \leq s < 2$. Consequently, when $1 \leq s < 2$ we can only assert that

$$\liminf_n \frac{S_n}{n^{1/s}} = \liminf_n \frac{ES_n}{n^{1/s}} \quad \text{a.s.}, \qquad \limsup_n \frac{S_n}{n^{1/s}} = \limsup_n \frac{ES_n}{n^{1/s}} \quad \text{a.s.} \tag{1.1}$$

This is a generalization of the Kolmogorov–Marcinkiewicz–Zygmund strong law of large numbers. Next, if $EX_n^2 < \infty, n \in \mathbb{N}$, it is not difficult to check that the Lindeberg–Feller central limit theorem becomes

$$\frac{S_n - ES_n}{\sqrt{\alpha_1(n)\sigma_1^2 + \cdots + \alpha_p(n)\sigma_p^2}} \xrightarrow{D} \Phi, \tag{1.2}$$

where Φ is the standard normal distribution function, and $\alpha_i(n) = \#\mathbb{N}_i \cap [1, n]$ and $\sigma_i^2 = \operatorname{Var} Y_i$ for $1 \leq i \leq p$. Finally, if $E|X_n| < \infty, n \in \mathbb{N}$, and τ is a finite $\{\sigma(X_1, \ldots, X_n)\}$-time with $E\tau < \infty$, then $E|S_\tau| < \infty$ and

$$\left(\min_{1 \leq i \leq p} EY_i\right) E\tau \leq ES_\tau \leq \left(\max_{1 \leq i \leq p} EY_i\right) E\tau. \tag{1.3}$$

This extension of Wald's equation can be obtained by arguing as in the classical i.i.d. case $p = 1$.

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Resting on these generalizations, we can now point out some basic facts concerning the renewal process $N(t)$, $t \geq 0$, defined as $N(t,\omega) = \max\{n : S_n(\omega) \leq t\}$, $\omega \in \Omega$. ($(\Omega, \mathcal{K}, P)$ is the underlying probability space.) Throughout we assume that $0 < EY_i = \mu_i$ and $EY_i^2 < \infty$, $1 \leq i \leq p$, and set $m = \min_{1\leq i\leq p} \mu_i$ and $M = \max_{1\leq i\leq p} \mu_i$. Let $\Omega_0 = \{\omega \in \Omega : (S_n(\omega) - ES_n)/n \to 0\}$. Then, on account of (1.1),

$$\liminf_n \frac{S_n(\omega)}{n} = \liminf_n \frac{ES_n}{n} = \liminf_n \frac{\alpha_1(n)\mu_1 + \cdots + \alpha_p(n)\mu_p}{n} \geq m > 0, \quad \omega \in \Omega_0,$$

and so $S_n(\omega) \to \infty, \omega \in \Omega_0$. This entails that $N(t,\omega) < \infty$ for all $t \geq 0$ and $\omega \in \Omega_0$. Moreover, since each X_n is finite-valued, $\lim_{t\to\infty} N(t,\omega) = \infty$, $\omega \in \Omega_0$. Further, using again (1.1), it follows that

$$M^{-1} \leq \liminf_{t\to\infty} \frac{N(t,\omega)}{t} \leq \limsup_{t\to\infty} \frac{N(t,\omega)}{t} \leq m^{-1}, \quad \omega \in \Omega_0.$$

The renewal function $U(t) = EN(t)$, $t \geq 0$, is finite-valued, and so right-continuous, also in this new set-up. To see this, choose $a > 0$ such that $b = \max_{1\leq i\leq p} P(Y_i \leq a) < 1$. Next, for $t \geq 0$, choose $l \in \mathbb{N}$ such that $al > t$. For any $j \geq 0$, we have

$$\begin{aligned} P(X_{j+1} + \cdots + X_{j+l} > t) &\geq P(X_{j+1} > a, \ldots, X_{j+l} > a) \\ &= (1 - P(X_{j+1} \leq a)) \ldots (1 - P(X_{j+l} \leq a)) > (1-b)^l. \end{aligned}$$

Therefore, if $kl \leq n < (k+1)l$, we get

$$P(S_n \leq t) \leq P(S_{kl} \leq t) \leq P(X_{j+1} + \cdots + X_{j+l} \leq t, 0 \leq j \leq k-1) \leq (1 - (1-b)^l)^k.$$

Hence, as $N(t) = \sum_{n\geq 1} I\{S_n \leq t\}$, we obtain

$$U(t) = \sum_{n\geq 1} P(S_n \leq t) = \sum_{k\geq 1} \sum_{kl\leq n<(k+1)l} \leq l \sum_{k\geq 1} (1 - (1-b)^l)^k < \infty.$$

Finally, by applying (1.3) and paralleling the proof of the i.i.d. case, it is not difficult to obtain the version of the elementary renewal theorem in the new setting, namely

$$M^{-1} \leq \liminf_{t\to\infty} \frac{U(t)}{t} \leq \limsup_{t\to\infty} \frac{U(t)}{t} \leq m^{-1}.$$

2. A central limit theorem

Takács (1956) established the following central limit theorem for the renewal process $N(t)$, $t \geq 0$, corresponding to the i.i.d. case, i.e. $p = 1$. If $\mu = EX_1$ and $\sigma^2 = \operatorname{Var} X_1$, then

$$\lim_{t\to\infty} P\left(\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^2}} \leq x \right) = \Phi(x), \quad x \in \mathbb{R}. \tag{2.1}$$

Formula (2.1) was generalized in several directions by various authors; in this connection, see the excellent summary of results on renewal theory in Gut (2009). The next example shows that, depending on the partition of $\mathbb{N}$ and additional conditions, an analogue of (2.1) holds in our setting.

Example 1. Suppose that $\alpha_i(n)/n \to r_i$, $1 \leq i \leq p$, and put $\mu = r_1\mu_1 + \cdots + r_p\mu_p$ and $\sigma^2 = r_1\sigma_1^2 + \cdots + r_p\sigma_p^2$. If $\alpha_1(n)\mu_1 + \cdots + \alpha_p(n)\mu_p - n\mu = o(\sqrt{n})$ as $n \to \infty$, then, resting on the proof of the classical i.i.d. case, one can see that (2.1) is verified.

Applying (1.2) and assuming that $\alpha_1(n)\sigma_1^2 + \cdots + \alpha_p(n)\sigma_p^2 \sim n\sigma^2$ as $n \to \infty$ for some $\sigma \in]0, \infty[$, we extend the central limit theorem to the new set-up. To do this, we need the following lemma.

Lemma 1. *Let f be the function whose graph is the polygonal line joining in order the points (ES_n, n), $n \geq 0$. Then, for any $c \in \mathbb{R}$,*

$$\lim_{t\to\infty} \frac{f(t)}{f\left(t + c\sqrt{f(t)}\right)} = 1. \tag{2.2}$$

Proof. Notice first that

$$\frac{t}{M} \leq f(t) \leq \frac{t}{m}, \quad t \geq 0, \tag{2.3}$$