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On a discrete version of the Laugesen–Morpurgo conjecture

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ABSTRACT

In this paper we use coupling arguments to prove a discrete 1-dimensional version of the Laugesen–Morpurgo conjecture. As an application, we derive a probabilistic proof of the 1-dimensional Laugesen–Morpurgo conjecture (for the 1-dimensional reflecting Brownian motion).

1. Introduction

The Laugesen–Morpurgo conjecture appeared, as we learned from Rodrigo Bañuelos, in connection with their work (see Laugesen and Morpurgo (1998)) on conformal extremals of the Riemann zeta function of eigenvalues. The conjecture states that the diagonal element of the Neumann heat kernel of the Laplacian in the unit ball $\mathbb{U} = \left\{x \in \mathbb{R}^2 : |x| < 1\right\}$ in $\mathbb{R}^2$ is a radially increasing function, that is

$$p(t, x, x) < p(t, y, y), \quad t \geq 0, \tag{1}$$

for all $x, y \in U$ with $0 \leq |x| < |y| \leq 1$, where $p(t, x, y)$ denotes the heat kernel for the Laplacian with Neumann boundary conditions (or, equivalently, the transition density for the Brownian motion with normal reflection on the boundary) in the unit disk U. The conjecture extends naturally to the Neumann heat kernel of the Laplacian in the unit ball $\mathbb{B} = \left\{x \in \mathbb{R}^d : ||x|| < 1\right\}$ in $\mathbb{R}^d$, $d \geq 1$.

The probabilistic interpretation of the conjecture is that a reflecting Brownian motion starting closer to the boundary is more likely to return to its starting position (after t units of time), than a reflecting Brownian motion starting further away from the boundary (after the same t units of time).

The physical interpretation is that introducing an atom of heat in a circular room with thermally insulated boundary, the closer this point to the boundary, the warmer we feel at this point, after any fixed number of units of time.

The intuition for this phenomenon is that starting closer to the boundary, the reflecting Brownian motion has a better chance of returning to this point since it will get more "push" directed towards this point, compared to a Brownian motion starting further away from the boundary. However, this is just an intuitive argument, and it is quite difficult to arrange it in a proof (despite the fact that it has been known for some time), the Laugesen–Morpurgo conjecture is still open at the present moment in its full generality (see Pascu and Pascu (2007) and Pascu and Pascu (2008) for some partial results towards the resolution of the conjecture).

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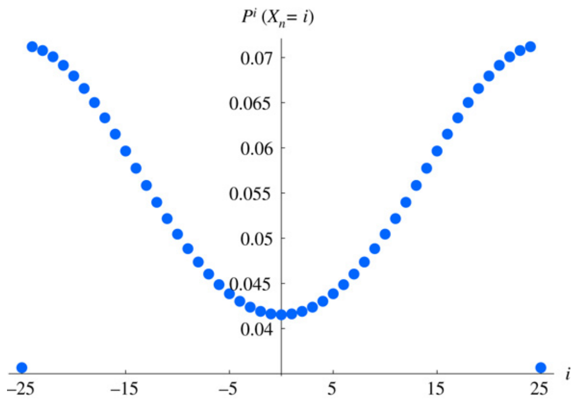

Fig. 1. The graph of the probabilities $P^i\,(X_n = i)$, $i = -s, \ldots, s$ for $s = 25$ and $n = 500$.

Recently, Banuelos et al. (in press) proved the following result related to the Laugesen–Morpurgo conjecture:

Theorem 1. *The diagonal element $p_{\mathbb{B}}(t, x, x)$ of the transition probabilities for the d-dimensional Bessel processes on $(0; 1]$, reflected at 1, is an increasing function of $x \in (0, 1]$ for $d > 2$ and this is false for $d = 2$.*

Remark 2. Since the norm of a d-dimensional Brownian motion is a Bessel process of order d, the above result is equivalent to the monotonicity with respect to $x \in (0, 1)$ (for any $t > 0$ arbitrarily fixed) of the integral mean

$$\int_0^{2\pi} p\left(t, x, xe^{i\theta}\right) d\theta$$

of the transition probabilities of the d-dimensional reflecting Brownian motion in the unit ball in $\mathbb{R}^d$.

In this paper we will prove a discrete 1-dimensional version of the Laugesen–Morpurgo conjecture, as follows: if X_n is a simple random walk on $\{-s, \ldots, s\}$ with reflecting barriers at $\pm s$, then for any $n \in \mathbb{N}$ arbitrarily fixed, $P^i\,(X_n = i)$ is a strictly increasing function of $|i|$, that is:

$$P^i\,(X_n = i) \le P^j(X_n = j), \tag{2}$$

for any $i, j \in \{-s + 1, \ldots, s - 1\}$ with $|i| < |j|$ and any $n \in \mathbb{N}$.

Trying to prove (2) by combinatorial methods (by explicit computation of the probabilities $P^i\,(X_n = i)$ for various i, n and s) is rather difficult, requiring the computation of the elements on the diagonal of a nth power of a certain $s \times s$ matrix.

It is interesting to note that the inequality (2) does not hold for $j = s$, as it can be seen from the Fig. 1. The reason for this is that

$$\begin{aligned} P^s\,(X_n = s) &= P^s\,(X_1 = s - 1)\,P^{s-1}\,(X_{n-1} = s) \\ &= P^{s-1}\,(X_{n-1} = s) \\ &= P^{s-1}\,(X_{n-2} = s - 1)\,P^{s-1}\,(X_1 = s) \\ &= \frac{1}{2} P^{s-1}\,(X_{n-2} = s - 1) \end{aligned}$$

which is smaller than $P^{s-1}\,(X_n = s - 1)$, since

$$\begin{aligned} P^{s-1}\,(X_n = s - 1) &= P^{s-1}\,(X_2 = s - 1)\,P^{s-1}\,(X_{n-2} = s - 1) + P^{s-1}\,(X_2 = s - 3)\,P^{s-3}\,(X_{n-2} = s - 1) \\ &= \frac{3}{4} P^{s-1}\,(X_{n-2} = s - 1) + \frac{1}{4} P^{s-3}\,(X_{n-2} = s - 1) \\ &> \frac{1}{2} P^{s-1}\,(X_{n-2} = s - 1)\,. \end{aligned}$$

Also note that when n is odd, (2) is trivial, since in this case $P^i\,(X_n = i) = 0$ for any $i \in \{-s, \ldots, s\}$.

Coupling methods are powerful tools which can be used to prove certain inequalities for the associated processes (see for example Pascu (2002) for a partial resolution of the Hot Spots conjecture of J. Rauch using coupling arguments). Our proof of the discrete version of Laugesen–Morpurgo conjecture for the 1-dimensional random walk on $\{-s, \ldots, s\}$ with reflecting barriers at s and $-s$ uses synchronous and mirror couplings of (reflecting) simple random walks, introduced in Section 3.

As an application of our main result (Theorem 15), we also derive a proof of the Laugesen–Morpurgo conjecture for the 1-dimensional Brownian motion, that is, we show that the inequality (1) holds for the transition density $p\,(t, x, y)$ of the 1-dimensional reflecting Brownian motion on the interval $(-1, 1)$ (see Corollary 21 and Remark 22).