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Forward–backward SDEs and the CIR model

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Abstract

We consider a forward–backward stochastic differential equation associated with the bond price for the Cox–Ingersoll–Ross interest rate model and prove an existence and uniqueness result. This technique is generalizable to multidimensional affine term structure models.

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1. Introduction

Cox et al. (1985) proposed a model for the short rate which, provided certain parametric restrictions are imposed, remains nonnegative and includes stochastic volatility. On the risk neutral probability space $(\Omega, \mathscr{F}, Q)$ assume that the short rate, X_t, has dynamics

$$\mathrm{d}X_t = \beta(\alpha - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}B_t$$

with $\alpha, \beta, \sigma > 0$ and $2\beta\alpha > \sigma^2$. The Cox–Ingersoll–Ross (CIR) model has been extensively studied and has lead to generalizations in several directions. Duffie and Kan (1996), Filipović (2001) and Boyle et al. (2002) study the CIR model and its generalizations and contain further references to the literature.

The price of the zero-coupon bond is given by

$$P(t,T) = E_Q\left[\exp\left(-\int_t^T X_u\,\mathrm{d}u\right)\middle|\mathscr{F}_t\right] \tag{1}$$

at time t for maturity T. For the CIR model the expectation in Eq. (1) can be solved in a number of different ways to obtain a well known closed-form solution for the bond price. The purpose of this paper is to provide an alternate approach, based on forward–backward stochastic differential equations (FBSDEs, for short), to deriving the bond price. This approach is generalizable beyond the one dimensional case of the CIR model to

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the multidimensional case of the general square-root affine factors process studied by Duffie and Kan (1996) and Elliott and van der Hoek (2001).

2. Stochastic flows and the forward measure

To begin we consider, as in Elliott and van der Hoek (2001), a stochastic flow associated with the short rate process

$$X_s^{t,x} = x + \int_t^s \beta(\alpha - X_v^{t,x})\,\mathrm{d}v + \int_t^s \sigma\sqrt{X_v^{t,x}}\,\mathrm{d}B_v$$

for all $0 \leqslant t \leqslant s \leqslant T$ and $x \in \mathbf{R}_{++}$. For all $t \in [0, T]$, since X_t is a Markov process, we have

$$P(t,T) = P(t,T,X_t), \tag{2}$$

Q—a.s., where we define

$$P(t,T,x) \triangleq E_Q\left[\exp\left(-\int_t^T X_v^{t,x}\,\mathrm{d}v\right)\right]. \tag{3}$$

Write $(\partial_x X_s^{t,x})$ for the derivative of $X_s^{t,x}$ with respect to the initial condition x. This derivative is well-defined and satisfies the linearized integral equation

$$(\partial_x X_s^{t,x}) = 1 - \beta\int_t^s (\partial_x X_v^{t,x})\,\mathrm{d}v + \frac{\sigma}{2}\int_t^s \frac{(\partial_x X_v^{t,x})}{\sqrt{X_v^{t,x}}}\,\mathrm{d}B_v. \tag{4}$$

Differentiating Eq. (3) we obtain

$$\partial_x P(t,T,x) = E_Q\left[\left(-\int_t^T \partial_x X_v^{t,x}\,\mathrm{d}v\right)\exp\left(-\int_t^T X_v^{t,x}\,\mathrm{d}v\right)\right]$$

subject to regularity conditions which allow the exchange of differentiation and integration. By considering the forward measure P^T, obtained by taking the zero-coupon bond price as numéraire, Elliott and van der Hoek (2001) show that

$$(\partial_x P(t,T,x))|_{x=X_t} = P(t,T,X_t)E_T[Y_t^{(B)}|\mathscr{F}_t], \tag{5}$$

where $E_T[]$ denotes expectation under P^T and $Y_t^{(B)} = (-\int_t^T(\partial_x X_v^{t,x})\,\mathrm{d}v)|_{x=X_t}$. Therefore, if the expectation under the forward measure of $(\partial_x X_v^{t,x})$ does not depend on x, the ordinary differential equation (5) can be solved to obtain an exponential affine form for the bond price. The derivation given in Elliott and van der Hoek (2001) proceeds by showing that the process

$$B_t^T \triangleq B_t - \int_0^t E_T[Y_u^{(B)}|\mathscr{F}_u]\sigma\sqrt{X_u}\,\mathrm{d}u \tag{6}$$

is a $\mathscr{F}_t$-Brownian motion with respect to the forward measure. Using Eq. (6) to write dynamics (4) with respect to the forward measure and the fact that, for all $t \leqslant v$, $X_v^{t,X_t} = X_v$ it can be shown that the conditional expectation $\hat{D}_{ts} \triangleq E_T[(\partial_x X_s^{t,X_t})|\mathscr{F}_t]$ satisfies

$$\hat{D}_{ts} = 1 - \beta\int_t^s \hat{D}_{tv}\,\mathrm{d}v + \frac{\sigma^2}{2}\int_t^s E_T[(\partial_x X_v^{t,x})|_{x=X_t}E_T[Y_v^{(B)}|\mathscr{F}_v]|\mathscr{F}_t]\,\mathrm{d}v \tag{7}$$

for $0 \leqslant t \leqslant s \leqslant T$, almost surely.

Further, using the semigroup (or flow) property, the chain rule, and elementary properties of conditional expectation Elliott and van der Hoek (2001) show that $\hat{D}_{ts}$ satisfies the integral equation

$$\hat{D}_{ts} = 1 - \beta\int_t^s \hat{D}_{tv_1}\,\mathrm{d}v_1 - \frac{\sigma^2}{2}\int_t^s\int_{v_1}^T \hat{D}_{tv_2}\,\mathrm{d}v_2\,\mathrm{d}v_1, \quad 0 \leqslant t \leqslant s \leqslant T. \tag{8}$$