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Martingale transforms between $\mathcal{Q}_1$ and $\mathcal{Q}_\Phi$ of martingale spaces

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ABSTRACT

The martingale transform techniques are used to show that given any Young function Φ with finite power p the corresponding $\mathcal{Q}_\Phi$ space is the martingale transform of $\mathcal{Q}_1$ and the converse result is also true. The results obtained here extend the corresponding results in the former literature.

1. Introduction

Martingale transforms were firstly introduced by Burkholder (1966) and since then have acquired intimate relation with some concepts in analysis (for example with conjugate harmonic functions), and a lot of applications in probability, in analysis, and even in Geometry (related to Banach spaces). Martingale transforms are especially useful in studying predictable martingale spaces. Characterizations of such spaces via martingale transforms are provided. So, naturally, many people have been attracted to this field. Garsia (1973) has given a satisfactory characterization about interchanging the space $\mathcal{D}_{p_1}$ to $\mathcal{D}_{p_2}$, or $H^s_{p_1}$ to $H^s_{p_2}$, via martingale transforms, for full scope index: $0 < p_1, p_2 < \infty$, $0 < p_1 < p_2 = \infty$. Then these results were improved by Chao and Long (1992). Ishak and Mogyorodi (1982) have obtained a characterization about interchanging the space $\mathcal{D}_1$ to $\mathcal{D}_\Phi$ via martingale transforms. This article is based on the work of Garsia (1973) and Ishak and Mogyorodi (1982). This article applies the martingale transform techniques to show that given any Young function Φ with finite power p the corresponding $\mathcal{Q}_\Phi$ space is the martingale transform of $\mathcal{Q}_1$ and the converse result is also true.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n\}_{n\geq 0}$ be a nondecreasing sequence of sub-σ-fields of $\mathcal{F}$, $f = (f_n)_{n\geq 0}$ be a scalar-valued martingale adapted to $\{\mathcal{F}_n\}_{n\geq 0}$. Denote $df = (df_n)_{n\geq 1}$ as the martingale differences with $df_n = f_n - f_{n-1}$, $n \geq 1$. For all $p : 0 < p < \infty$, define the square function and the conditioned square function of f as follows:

$$S_n(f) = \left(\sum_{i=1}^{n} |df_i|^2\right)^{\frac{1}{2}}, \qquad S(f) = \sup_{n\geq 1} S_n(f).$$

$$s_n(f) = \left(\sum_{i=1}^{n} E\left(|df_i|^2 | \mathcal{F}_{i-1}\right)\right)^{\frac{1}{2}}, \qquad s(f) = \sup_{n\geq 1} s_n(f).$$

Let $\Phi(x)$ be an increasing convex function on $[0, +\infty)$ with $\Phi(0) = 0$. It is well known that any convex function like this has the form

$$\Phi(x) = \int_0^x \varphi(t)\mathrm{d}t, \quad \forall x > 0$$

where $\varphi(x)$ is a positive, increasing, finite-valued and left continuous function on $[0, +\infty)$.

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If $\lim_{x\to\infty} x^{-1}\Phi(x) = \infty$, then $\Phi(x)$ is called a Young function.

Define the inverse of $\varphi(t)$

$$\psi(s) = \inf\{t : \varphi(t) \geq s\}$$

with the convention: $\inf\{\emptyset\} = +\infty$. It is easy to see that $\psi(s)$ is still of the same kind. Its integral $\Psi(x) = \int_0^x \psi(t)\mathrm{d}t$, is a generalized convex function, and will be called Young's complementary function of Φ.

We suppose that the power

$$p = \sup_{x>0} \frac{x\varphi(x)}{\Phi(x)}$$

of Φ is finite. This implies that the inverse function Φ^{-1} of Φ exists and has the form

$$\Phi^{-1}(x) = \int_0^x m(t)\mathrm{d}t.$$

Here $m(t)$ is a decreasing function and we can easily see that

$$m(t) = \frac{1}{\varphi(\Phi^{-1}(t))}, \quad t > 0.$$

To show these we remark that if p is finite then neither $\Phi(x)$ nor $\varphi(x)$ do vanish for $x > 0$. Consequently, $\Phi(x)$ strictly increases. This and the continuity of Φ imply that inverse function Φ^{-1} exists, it is concave and is of the above integral form.

For any Young function $\Phi(x)$, the Orlicz space $L^\Phi = L^\Phi(\Omega, \mathcal{F}, P)$ of scalar-valued functions is defined by

$$L^\Phi = \{f : \|f\|_{L^\Phi} < \infty\}, \|f\|_{L^\Phi} = \inf\{a > 0 : E\left[\Phi(a^{-1}|f|)\right] \leq 1\}.$$

$\|\cdot\|_{L^\Phi}$ is a norm on L^Φ. The normed vector space L^Φ is complete.

These definitions and results are due to Long (1993) and Ishak and Mogyorodi (1982).

Let Γ be the class of non-negative, nondecreasing and adapted sequences $\lambda = (\lambda_n)_{n\geq 0}$ with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ and $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. Now define Hardy spaces and Hardy–Orlicz spaces of martingales as follows:

$$H_p^s = \{f = (f_n)_{n\geq 0} : s(f) \in L_p\}, \|f\|_{H_p^s} = \|s(f)\|_{L_p}.$$

$$\mathcal{D}_p = \{f = (f_n)_{n\geq 0} : \exists\{\lambda_n\} \in \Gamma, s.t. |f_n| \leq \lambda_{n-1}, a.e., \lambda_\infty \in L_p\},$$

$$\|f\|_{\mathcal{D}_p} = \inf_{\{\lambda_n\}\in\Gamma} \|\lambda_\infty\|_{L_p}.$$

$$\mathcal{D}_\Phi = \{f = (f_n)_{n\geq 0} : \exists\{\lambda_n\} \in \Gamma, s.t. |f_n| \leq \lambda_{n-1}, a.e., \lambda_\infty \in L^\Phi\},$$

$$\|f\|_{\mathcal{D}_\Phi} = \inf_{\{\lambda_n\}\in\Gamma} \|\lambda_\infty\|_{L^\Phi}.$$

$$\mathcal{Q}_p = \{f = (f_n)_{n\geq 0} : \exists\{\lambda_n\} \in \Gamma, s.t. S_n(f) \leq \lambda_{n-1}, a.e., \lambda_\infty \in L_p\},$$

$$\|f\|_{\mathcal{Q}_p} = \inf_{\{\lambda_n\}\in\Gamma} \|\lambda_\infty\|_{L_p}.$$

$$\mathcal{Q}_\Phi = \{f = (f_n)_{n\geq 0} : \exists\{\lambda_n\} \in \Gamma, s.t. S_n(f) \leq \lambda_{n-1}, a.e., \lambda_\infty \in L^\Phi\},$$

$$\|f\|_{\mathcal{Q}_\Phi} = \inf_{\{\lambda_n\}\in\Gamma} \|\lambda_\infty\|_{L^\Phi}.$$

The martingales of $\mathcal{D}_1$ and $\mathcal{D}_\Phi$ spaces are respectively said to be the predictable martingales in L_1 and L^Φ. For some further results on the above spaces see Long (1993) and Weisz (1994). Ishak and Mogyorodi (1982) have obtained a characterization about interchanging the space $\mathcal{D}_1$ to $\mathcal{D}_\Phi$. On the basis of this, the relations between the $\mathcal{Q}_1$-space and the $\mathcal{Q}_\Phi$-space are investigated.

Remark 1. Infimum in the definitions of $\|\cdot\|_{\mathcal{Q}_p}$ and $\|\cdot\|_{\mathcal{Q}_\Phi}$ is achieved. In fact, let $\{\lambda_n^{(k)}\}$ be a predictable sequence of $\{S_n(f)\}$ for every $k \in \mathbb{N}$ such that $\|\lambda_\infty^{(k)}\|_{L_p} \to \|f\|_{\mathcal{Q}_p} (k \to +\infty)$. Then $\{\lambda_n^{(f)}\}_{n\geq 0}$ with $\lambda_n^{(f)} = \inf_k \lambda_n^{(k)}$, is a predictable sequence of $\{S_n(f)\}$ satisfying $\|f\|_{\mathcal{Q}_p} = \|\lambda_\infty^{(f)}\|_{L_p}$, which will be called the optimal L_p—predicting quadratic sequences for f. The proof for $\mathcal{Q}_\Phi$ is similar.

Definition 1. Let $f = (f_n)_{n\geq 0}$ be a martingale and $v = (v_n)_{n\geq 0}$ be an adapted process.The following transform

$$g_n = \sum_{i=1}^n v_{i-1} df_i, \quad n \geq 1, \qquad g_0 = 0, \ a.e.$$

is called a martingale transform.

Lemma 1. *Let $f = (f_n)_{n\geq 0}$ be a martingale. Then f converges a.e. on the set $\{x : s(f) < \infty\}$.*

Remark 2. It is well known that for any martingale $f = (f_n)_{n\geq 0}$ with $E(Md) < \infty$, $Md = \sup |df_n|$.

$$\{x : f \text{ converges}\} = \{x : Mf < \infty\} = \{x : S(f) < \infty\} \quad (\text{modulo measure } 0).$$