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Unbiasedness on rays of the tests of locations of homogeneous means in isotonic regression

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Abstract

The location of a normal mean can be specified by a vector or by an ellipsoid. In isotonic regression the mean must be from an order restricted cone. For the likelihood ratio tests (LRTs) for the locations of an isotonic mean a simple condition is derived for the tests to be unbiased on a ray. Based on this result the most favorable point within the isotonic regression specification and the set that contains the least favorable points within the null hypothesis are identified. It is also shown that when the null hypothesis points to a homogeneous vector, the LRTs are unbiased.

Keywords: Isotonic regression; Likelihood ratio test; Order restricted inference; Power functions; Projection

1. Introduction

Let $X^{(1)}, \ldots, X^{(n)}$ be a random sample from a k-variate normal population with mean μ and covariance matrix $\sigma^2 V$. A confidence region of μ is of the form of an ellipsoid

$$B = \{x \in R^k : (x-u)' V^{-1} (x-u) \leqslant d^2\}.$$

Thus when u is given, $\mathrm{H} : \mu \in B$ is a location hypothesis with $\mathrm{H} : \mu = u$ being a special case for $d = 0$. Often this hypothesis with $u = c1_k = (c, \ldots, c)'$ is of interest. Repeated measure experiments in which the components of μ are treatment means and c is a known control mean provide an example. With the norm induced from the inner product $\langle x, y \rangle = x' V^{-1} y$ we can write

$$B = \{x \in R^k : \|x - u\| \leqslant d\}, \quad u = c1_k, \ d \geqslant 0. \tag{1}$$

Here the notation B is from the fact that the ellipsoid is now a ball, or from the fact that (1) in essence is a bound restriction on the norm. In isotonic regression the mean must be from an order restricted cone

$$C = \{x = (x_1, \ldots, x_k)' : x_1, \ldots, x_k \text{ satisfy given partial ordering}\}. \tag{2}$$

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Thus $\mathrm{H}: \mu \in B \cap C$ is a location hypothesis in isotonic regression. In this paper we study the behavior of power functions of LRTs of H_0 versus H_1 where

$$\mathrm{H}_0 = B \cap C, \quad \mathrm{H}_1 = C \tag{3}$$

in σ^2 known and unknown cases. For convenience testing H_0 versus H_1 means testing H_0 versus $\mathrm{H}_1 - \mathrm{H}_0$.

Among many tests that have been extensively studied are that of $\mu \in C$ versus $\mu \in R^k$ called *CL* type tests, and that of $\mu_1 = \cdots = \mu_k$ versus $\mu \in C$ called *LC* type tests since both R^k and $\{\mu : \mu_1 = \cdots = \mu_k\}$ are *L*, linear spaces, while *C* is a cone. The *LC* type LRTs are unbiased, see Hu and Wright (1994). It is argued in Robertson et al. (1988) that the LRT is biased for some *CL* tests. For *CC* type tests Warrack and Robertson (1984), Menendez and Salvador (1991) find that when the two cones are oblique, LRTs are inadmissible. The type of hypotheses considered in this paper is none of the above since $B \cap C$ is not a linear space nor a cone. It is similar to that in Hu (1999) where the location of μ is rendered as componentwise bound restrictions and the tests are proven unbiased using the limit of truncated isotonic regressions. Sasabuchi et al. (2003) propose tests for homogeneity of multivariate normal mean under an order restriction where the covariance matrix is unknown.

Our study is based on a concept called unbiasedness on rays. For H_0 and H_1 in (3) let

$$r = \{\mu_0 + tv_0 : t \geqslant 0\}, \quad \mu_0 \in \mathrm{H}_0, \; 0 \neq v_0 \in \mathrm{H}_1. \tag{4}$$

Then r is a ray originated at μ_0 in H_0 going in the direction of v_0 in H_1. This ray lies entirely in H_1. A test of H_0 versus H_1 is said to be unbiased on r if its power function is nondecreasing on r. Clearly if a test is unbiased on each ray in (4), then it is an unbiased test.

We obtain a simple sufficient condition for a test to be unbiased on ray (4). The implications include that the LRTs in σ^2 known and unknown cases are unbiased when $d = 0$. Generally for $d \geqslant 0$ we show that $u = c1_k$ is the most favorable point within C and the portion of the shell of B inside C contains the least favorable points within H_0. The exact locations of the least favorable points however are ordering dependent.

2. Tests and powers

Denote the projection of x onto a closed, convex set G, the vector in G that minimizes the distance to x over all vectors in G, by $P(x|G)$. By conventional discussion we obtain the following theorem.

Theorem 1. *For* H_0 *and* H_1 *in* (3), *let*

$$T_1 = \|P(\overline{X}|\mathrm{H}_0) - P(\overline{X}|\mathrm{H}_1)\|^2$$

and

$$T_2 = \frac{\|P(\overline{X}|\mathrm{H}_0) - P(\overline{X}|\mathrm{H}_1)\|^2}{\|\overline{X} - P(\overline{X}|\mathrm{H}_1)\|^2 + \frac{1}{n}\sum_{i=1}^n \|X^{(i)} - \overline{X}\|^2}.$$

Then T_1 *is a* LRT *statistic when* σ^2 *is known*, T_2 *is a* LRT *statistic when* σ^2 *is unknown*, *and* H_0 *is rejected for large values of the test statistics*.

For $a_1 > 0$ and $a_2 > 0$ define events $A_1 = \{\|P(\overline{X}|\mathrm{H}_0) - P(\overline{X}|\mathrm{H}_1)\|^2 \leqslant a_1\}$ and $A_2 = \{\|P(\overline{X}|\mathrm{H}_0) - P(\overline{X}|\mathrm{H}_1)\|^2 \leqslant a_2\|\overline{X} - P(\overline{X}|\mathrm{H}_1)\|^2 + \frac{a_2}{n}\sum_{i=1}^n \|X^{(i)} - \overline{X}\|^2\}$. Clearly if $\Pr(A_1)$ is a nonincreasing function of μ on ray (4), then the LRT is unbiased on that ray when σ^2 is known. If $\Pr(A_2)$ is a nonincreasing function of μ on ray (4), then the LRT is unbiased on that ray when σ^2 is unknown. Define

$$A = \{x \in R^k : \|P(x|\mathrm{H}_0) - P(x|\mathrm{H}_1)\|^2 \leqslant a\|x - P(x|\mathrm{H}_1)\|^2 + b\}. \tag{5}$$

Then $\Pr(A_2) = E[\Pr(\overline{X} \in A)|a = a_2, b = a_2 Y]$ with $Y = \sum_{i=1}^n \|X^{(i)} - \overline{X}\|^2/n$ since Y is independent of $\overline{X}$ and its distribution is free of μ. Also note that $\Pr(A_1) = \Pr(\overline{X} \in A)$ with $a = 0$, $b = a_1$. Therefore if $\Pr(\overline{X} \in A)$ is a nonincreasing function of μ on ray (4) for all $a \geqslant 0$ and for all $b \geqslant 0$, then the LRTs in σ^2 known and unknown cases are both unbiased on that ray.