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Approximate predictive pivots for autoregressive processes

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ABSTRACT

In this paper the author considers an autoregressive process where the parameters of the process are unknown and try to obtain pivots for predicting future observations. If we do a probabilistic prediction with the estimated model, where the parameters are estimated by a sample of size n, we introduce an error of order n^{-1} in the coverage probabilities of the prediction intervals. However we can reduce the order of the error if we calibrate adequately the estimated prediction bounds. The solution obtained can be expressed in terms of an approximate predictive pivot.

1. Introduction

The general setting is of prediction of an absolutely continuous (a.c.) random variable Z based on the observation $y = (y_1, y_2, \ldots, y_n)$ corresponding to a random vector $Y = (Y_1, Y_2, \ldots, Y_n)$, where the laws of Y and Z depend on a common and unknown parameter $\theta \in \Theta \subset \mathbf{R}^d$. A prediction statement about Z is often given by prediction limits, i.e. real functions $K_\alpha(\cdot)$ such that

$$P_\theta\{Z \leq K_\alpha(Y)\} = \alpha,$$

for every $\theta \in \Theta$ and for any fixed $\alpha \in (0, 1)$. The above probability is usually called coverage probability and it is calculated with respect to the joint density of Z and Y. Sometimes the existence of exact (predictive) pivotal quantities, that is of functions of Z and Y whose distribution does not depend on θ, permit us to find an exact solution. But this is the exception. Here we look for *approximate* prediction limits and predictive pivots. An approximate solution is to take $K_\alpha(Y) = q_\alpha(\hat{\theta})$, where $q_\alpha(\theta)$ is the α-quantile of the conditional distribution of Z given $Y = y$, that we also assume absolutely continuous, and $\hat{\theta}$ is an efficient estimator of θ. Note that, if we denote the conditional density of Z given $Y = y$, $g(z; \theta|y)$, then $q_\alpha(\hat{\theta})$ will be the α-quantile of the so-called estimative density $g(z; \hat{\theta}|y)$. However these predictions limits are usually imprecise, having coverage error of order $O(n^{-1})$, that is

$$P_\theta\{Z \leq q_\alpha(\hat{\theta})\} = \alpha + O(n^{-1}).$$

This is a well known result; indeed Barndorff-Nielsen and Cox (1996) suggest a way to correct these quantiles obtaining prediction limits with a coverage error of order $o(n^{-1})$. The solution can be expressed in terms of a *predictive* density whose quantiles are precisely these predictions bounds. We will apply this method to the case where $Y = (Y_1, Y_2, \ldots, Y_n)$ is such that

$$Y_{k+1} - \mu = \sum_{j=1}^{p} \phi_j (Y_{k-j+1} - \mu) + \varepsilon_{k+1}, \quad k \in \mathbb{Z},$$

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where the innovation $\varepsilon_{k+1} \sim N(0, \sigma^2)$, $\phi_1, \ldots, \phi_p$ are the coefficients of a stationary autoregressive process of order p, $AR(p)$ for short, and where $Y_0, Y_{-1}, \ldots, Y_{-p+1}$ are assumed to be known and fixed. All the parameters, $\mu, \phi_1, \ldots, \phi_p$ and σ are unknown and $Z = Y_{n+1}$,

The paper is organized in the following way. In the next section we give the details of the Barndorff-Nielsen and Cox method. Then we solve the $AR(p)$ case. Some simulations are made in order to study how the corrections work.

2. The method

In this section we explain, in a concise and heuristic manner, the method of Barndorff-Nielsen and Cox (1996) to modify the estimative density $g(z; \hat{\theta}|y)$ to get better, asymptotically, prediction intervals or limits and how to obtain approximate predictive pivots. For more details, we refer the reader to that paper.

When we use $g(z; \hat{\theta}|y)$ instead of $g(z; \theta|y)$, we are introducing the error in the estimation of θ; we can correct this effect if we take into account the incertitude in $\hat{\theta}$ about θ. We know, by the *conditionality principle* that, when estimating θ, we should consider the conditional distribution of $\hat{\theta}$ given $A(Y) = a$, where A is an *ancillary* statistics (distribution free of θ and such that $(\hat{\theta}, A)$ is a minimal sufficient statistics for θ). Then, if we have a minimal predictive sufficient reduction of the data, W, we take W instead of Y, and if there is a decomposition $W = (\hat{\theta}, A)$ where A has a distribution free of θ, or approximately free (here we use the *continuity principle)*, we shall suppose A is fixed when we study long-run properties.

We shall omit A in the notation, whenever this does not produce confusion. Consider now that the joint law of (Z, Y) is absolutely continuous, with respect to the Lebesgue measure. Then, to predict Z we look for certain functions $K^{(\alpha)}(\hat{\theta})$ such that,

$$P_\theta\{Z \le K^{(\alpha)}(\hat{\theta})|A = a\} = \alpha \quad \forall \theta \in \Theta, \alpha \in (0, 1), \tag{1}$$

up to order $o_p(n^{-1})$.

Let $G(z; \theta|\hat{\theta}) = P_\theta\{Z \le z|W = (\hat{\theta}, a)\}$, that is the distribution function of Z given $Y = y$ (note that W is transitive) considered as function of $z, \theta, \hat{\theta}$ and a; and $g(z; \theta|\hat{\theta})$ the corresponding density function. Note that $P_\theta\{Z \le z|A = a\} = E_\theta(G(z; \theta|\hat{\theta})|A = a)$. Let $q^{(\alpha)}(\theta)$ be such that

$$G(q^{(\alpha)}(\theta); \theta|\theta) = \alpha, \quad \forall \theta \in \Theta.$$

Suppose we can write

$$E_\theta(G(q^{(\alpha)}(\hat{\theta}); \theta|\hat{\theta})) = \alpha + Q(q^{(\alpha)}(\theta), \theta) + o_p(n^{-1}),$$

where $Q(q^{(\alpha)}(\theta), \theta) = \frac{b(q^{(\alpha)}(\theta),\theta)}{n}$, and $b(\cdot, \cdot)$ is a smooth function, then

$$\begin{aligned}\int_{-\infty}^{q^{(\alpha)}(\hat{\theta}) - \frac{Q(q^{(\alpha)}(\theta),\theta)}{g(q^{(\alpha)}(\hat{\theta});\theta|\hat{\theta})}} g(z; \theta|\hat{\theta})\mathrm{d}z &= G(q^{(\alpha)}(\hat{\theta}); \theta|\hat{\theta}) - \int_{q^{(\alpha)}(\hat{\theta}) - \frac{Q(q^{(\alpha)}(\theta),\theta)}{g(q^{(\alpha)}(\hat{\theta});\theta|\hat{\theta})}}^{q^{(\alpha)}(\hat{\theta})} g(z; \theta|\hat{\theta})\mathrm{d}z \\ &= G(q^{(\alpha)}(\hat{\theta}); \theta|\hat{\theta}) - Q(q^{(\alpha)}(\theta), \theta) + o_p(n^{-1}),\end{aligned}$$

where we assume that, conditioning on $A = a$, $\hat{\theta} - \theta = O_p(n^{-1/2})$. Then by taking the expectation with respect to $\hat{\theta}$ we obtain that

$$K^{(\alpha)}(\hat{\theta}) = q^{(\alpha)}(\hat{\theta}) - \frac{Q(q^{(\alpha)}(\theta), \theta)}{g(q^{(\alpha)}(\hat{\theta}); \theta|\hat{\theta})} = q^{(\alpha)}(\hat{\theta}) - \frac{Q(q^{(\alpha)}(\hat{\theta}), \hat{\theta})}{g(q^{(\alpha)}(\hat{\theta}); \hat{\theta}|\hat{\theta})} + o_p(n^{-1}),$$

is a solution of (1). Also we could get a predictive density, denoted by $\hat{p}(z|y)$, with these quantiles up to order n^{-1}, that is

$$\int_{-\infty}^{K^{(\alpha)}(\hat{\theta})} \hat{p}(z|y)\mathrm{d}z = \alpha + o_p(n^{-1}).$$

To obtain it, consider the identity

$$\int_{-\infty}^{q^{(\alpha)}(\hat{\theta})} g(u; \hat{\theta}|\hat{\theta})\mathrm{d}u = \alpha,$$

by doing the change of variable $u = z + \frac{Q(z;\hat{\theta})}{g(z;\hat{\theta}|\hat{\theta})}$ we obtain that

$$\int_{-\infty}^{z_0} (1 + \partial_z \hat{r}) g(z + \hat{r}; \hat{\theta}|\hat{\theta})\mathrm{d}z = \alpha,$$

where

$$\hat{r} = \frac{Q(z; \hat{\theta})}{g(z; \hat{\theta}|\hat{\theta})}. \quad \text{and} \quad z_0 = K^{(\alpha)}(\hat{\theta}) + o_p(n^{-1}). \tag{2}$$