Statistics and Probability Letters 78 (2008) 2751–2755

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On the connections between weakly stable and pseudo-isotropic distributions

B.H. Jasiulis [a,*], J.K. Misiewicz [b,1]

[a] Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

[b] Department of Mathematics Informatics and Econometrics, University of Zielona Góra, ul. Podgórna 50, 65-246, Zielona Góra, Poland

ARTICLE INFO

Article history:
Received 5 June 2007
Received in revised form 15 February 2008
Accepted 14 March 2008
Available online 23 March 2008

MSC:
60A10
60B05
60E05
60E07
60E10

ABSTRACT

A random vector $\mathbf{X}$ is weakly stable iff for all $a, b \in \mathbb{R}$ there exists a random variable Θ such that $a\mathbf{X}+b\mathbf{X}' \stackrel{d}{=} \mathbf{X}\Theta$. This is equivalent (see Misiewicz et al. [Misiewicz, J.K., Oleszkiewicz, K., Urbanik, K., 2005. Classes of measures closed under mixing and convolution. Weak stability. Stud. Math. 167 (3), 195–213]) to the condition that for all random variables Q_1, Q_2 there exists a random variable Θ such that

$$\mathbf{X}Q_1 + \mathbf{X}'Q_2 \stackrel{d}{=} \mathbf{X}\Theta, \qquad (*)$$

where $\mathbf{X}, \mathbf{X}', Q_1, Q_2, \Theta$ are independent. Some of the weakly stable distributions turn out to be the extreme points for the class of pseudo-isotropic distributions, where the distribution is pseudo-isotropic if all its one-dimensional projections are the same up to a scale parameter. We show here that the scaling function for pseudo-isotropic distribution can define a generalized distribution iff it is an α-norm for some $\alpha > 0$.

1. Introduction

The idea of generalized convolutions was studied by K. Urbanik in a series of papers, see Urbanik (1964, 1973, 1984, 1986, 1988). This area turned out to be interesting and it has been developed extensively till now.

In the 1970s, a series of papers appeared written mainly by Kucharczak and Urbanik (see e.g. Kucharczak and Urbanik (1974) and Urbanik (1964)), where the authors introduced the idea of weakly stable distributions on $[0, \infty)$. More precisely, a probability measure μ on $[0, \infty)$ was called weakly stable if

$$\forall\, a, b > 0 \exists \lambda \quad \text{on } [0, \infty) \quad T_a\mu * T_b\mu = \mu \circ \lambda,$$

where $(T_a\mu)(A) = \mu(A/a)$, $(\mu \circ \lambda)(A) = \int \mu(A/s)\lambda(ds)$ for every Borel set A in $[0, \infty)$. Currently measures μ for which this condition holds are rather called $\mathbb{R}_+$-weakly stable. They are extensively studied, see e.g. Urbanik (1986) and Vol'kovich (1992, 1985, 1984).

Recently Misiewicz, Oleszkiewicz and Urbanik (see Misiewicz et al. (2005)) gave a definition and showed some basic properties of weakly stable distributions on $\mathbb{R}^n$, or on a separable Banach spaces $\mathbb{E}$. The definition of weakly stable distributions is almost the same as the one given by Kucharczak and Urbanik, except for the fact that constants a, b can be any real numbers. The authors obtained in Misiewicz et al. (2005) the full characterization of weakly stable distributions with non-trivial discrete part. In Misiewicz (2006) Misiewicz studied basic properties of the weak generalized convolution based on weakly stable distribution.

* Corresponding author.
E-mail addresses: jasiulis@math.uni.wroc.pl (B.H. Jasiulis), j.misiewicz@wmie.uz.zgora.pl (J.K. Misiewicz).

1 This paper was written while the second author was a visiting professor of Delft Institute of Applied Mathematics, Delft University of Technology, Holland.

0167-7152/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2008.03.016

In this paper we present some new results on weak generalized convolutions and their connections with pseudo-isotropic distributions, i.e. multidimensional symmetric distributions having all one-dimensional margins the same up to a scale parameter. We show that pseudo-isotropy defines a weak generalized convolution iff the corresponding scale function is the ℓ_p norm on $\mathbb{R}^n$. This result links the geometry of L_p-spaces with functional equations, theory of pseudo-isotropic distributions and generalized convolutions.

Throughout this paper we denote by $\mathcal{L}(\mathbf{X})$ the distribution of the random vector $\mathbf{X}$. If random vectors $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution we will write $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$. By $\mathcal{P}(\mathbb{E})$ we denote the set of all probability measures on a separable Banach space (or on a set) $\mathbb{E}$. We will use the simplified notation $\mathcal{P}(\mathbb{R}) = \mathcal{P}$, $\mathcal{P}([0, +\infty)) = \mathcal{P}^+$. For every $a \in \mathbb{R}$ and every probability measure μ we define the rescaling operator $T_a : \mathcal{P}(\mathbb{E}) \to \mathcal{P}(\mathbb{E})$ as follows

$$T_a\mu(A) = \begin{cases} \mu(A/a) & \text{for } a \neq 0; \\ \delta_0(A) & \text{for } a = 0, \end{cases}$$

for every Borel set $A \in \mathbb{E}$. Equivalently $T_a\mu$ is the distribution of the vector $a\mathbf{X}$ if μ is the distribution of the vector $\mathbf{X}$. The scale mixture $\mu \circ \lambda$ of a measure $\mu \in \mathcal{P}(\mathbb{E})$ with respect to the measure $\lambda \in \mathcal{P}$ is defined by:

$$\mu \circ \lambda(A) \stackrel{\text{def}}{=} \int_{\mathbb{R}} T_s\mu\,(A)\,\lambda(\mathrm{d}s).$$

It is easy to see that $\mu \circ \lambda$ is the distribution of the random vector $\mathbf{X}\Theta$ if $\mu = \mathcal{L}(\mathbf{X})$, $\lambda = \mathcal{L}(\Theta)$, $\mathbf{X}$ and Θ are independent. In the language of characteristic functions we obtain

$$\widehat{\mu \circ \lambda}(\mathbf{t}) = \int_{\mathbb{R}} \widehat{\mu}(\mathbf{t}s)\lambda(\mathrm{d}s).$$

Notice that for a symmetric random vector $\mathbf{X}$ independent of random variable Θ we have $\mathbf{X}\Theta \stackrel{d}{=} \mathbf{X}|\Theta|$. From this property we obtain that if μ is a symmetric probability distribution then for $|\lambda| = \mathcal{L}(|\Theta|)$ we have

$$\mu \circ \lambda = \mu \circ |\lambda|.$$

Definition 1. A probability measure $\mu \in \mathcal{P}(\mathbb{E})$ is weakly stable (or $\mathbb{R}_+$-weakly stable) if for every choice of $\lambda_1, \lambda_2 \in \mathcal{P}$ ($\lambda_1, \lambda_2 \in \mathcal{P}_+$) there exists $\lambda \in \mathcal{P}$ ($\lambda \in \mathcal{P}_+$) such that

$$(\lambda_1 \circ \mu) * (\lambda_2 \circ \mu) = \lambda \circ \mu.$$

If μ is not symmetric then the measure λ is uniquely determined. This fact was proven in Misiewicz et al. (2005) for a weakly stable measure μ, and in Urbanik (1976) for a $\mathbb{R}_+$-weakly stable measure μ. If the measure μ is symmetric then only the symmetrization of λ is uniquely determined (see Misiewicz et al. (2005, Remark 1)). In this case we can always replace the measure λ by its symmetrization $(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}) \circ \lambda$. For convenience in this paper we assume that for symmetric μ the measure λ is concentrated on $[0, \infty)$ taking if necessary $|\lambda|$ instead of λ.

2. Generalized weak convolution

Definition 2. Let $\mu \in \mathcal{P}(\mathbb{E})$ be a nontrivial weakly stable measure, and let λ_1, λ_2 be probability measures on $\mathbb{R}$. If

$$(\lambda_1 \circ \mu) * (\lambda_2 \circ \mu) = \lambda \circ \mu,$$

then the weak generalized convolution of the measures λ_1, λ_2 with respect to the measure μ (notation $\lambda_1 \oplus_\mu \lambda_2$) is defined as follows

$$\lambda_1 \oplus_\mu \lambda_2 = \begin{cases} \lambda & \text{if } \mu \text{ is not symmetric;} \\ |\lambda| & \text{if } \mu \text{ is symmetric.} \end{cases}$$

In the case of symmetric weakly stable distribution, μ, we take $\lambda_1 \oplus_\mu \lambda_2 = |\lambda|$ in order to get uniqueness of this measure. If Θ_1, Θ_2 are random variables with distributions λ_1, λ_2 respectively then the random variable with distribution $\lambda_1 \oplus_\mu \lambda_2$ will be denoted as $\Theta_1 \oplus_\mu \Theta_2$. Thus we have

$$\Theta_1\mathbf{X}' + \Theta_2\mathbf{X}'' \stackrel{d}{=} (\Theta_1 \oplus_\mu \Theta_2)\,\mathbf{X},$$

where $\mathbf{X}, \mathbf{X}', \mathbf{X}''$ have distribution μ, $\Theta_1, \Theta_2, \mathbf{X}', \mathbf{X}''$ and $\Theta_1 \oplus_\mu \Theta_2, \mathbf{X}$ are independent. Now it is easy to see that the following lemma holds.

Lemma 1. *If the weakly stable measure $\mu \in \mathcal{P}(\mathbb{E})$ is not trivial then $\oplus_\mu$ is commutative and associative. Moreover for all $\lambda_1, \lambda_2, \lambda \in \mathcal{P}$ $\lambda_1 \oplus_\mu \lambda_2$ is uniquely determined and the following conditions hold:*

(i) $\lambda \oplus_\mu \delta_0 = \lambda$ *(*$\lambda \oplus_\mu \delta_0 = |\lambda|$ *if* μ *is symmetric);*

(ii) $T_a(\lambda_1 \oplus_\mu \lambda_2) = (T_a\lambda_1) \oplus_\mu (T_a\lambda_2)$*;*