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Statistics & Probability Letters 77 (2007) 632–643
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

Self-normalized Wittmann's laws of iterated logarithm in Banach space

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Received 8 August 2006; accepted 20 September 2006
Available online 18 October 2006

Abstract

For a sequence of independent symmetric Banach space valued random variables $\{X_n, n \geqslant 1\}$, we obtain the self-normalized Wittmann's law of iterated logarithm (LIL) and give the upper bound for the non-random constant.

MSC: 60F05; 60B12; 60F99

Keywords: Banach space; Law of iterated logarithm; Rademacher series; Self-normalizer; Symmetric random variables

1. Introduction

Let B be a real separable Banach space with $\|\cdot\|$ and topological dual B^*. For a B-valued random variable X and some $p \in [1, \infty)$, we write $X \in WM_0^p$ if for all $f \in B^*$, we have $Ef(X) = 0$ and $E|f(X)|^p < \infty$. Throughout $\{X_n, n \geqslant 1\}$ is a sequence of independent B-valued random variables defined on a probability space (Ω, F, P) and $\{\varepsilon_n, n \geqslant 1\}$ is an independent Rademacher series supported on the same probability space (Ω, F, P) and independent of $\{X_n, n \geqslant 1\}$. For each $n \geqslant 1$, put $S_n = \sum_{i=1}^n X_i$, $W_{n,p} = \sup_{f \in B_1^*}(\sum_{i=1}^n |f(X_i)|^p)^{1/p}$ where B_1^* is the unit ball of B^*. As usual, $L_2 x$ denotes the function $\log \log \max\{e^e, x\}$.

Griffin and Kuelbs (1989, 1991) established some extensions of the law of iterated logarithm (LIL) via self-normalizations for independent real valued random variables both in the symmetric and non-symmetric cases. However, many of these results require a symmetry assumption. Afterwards, Godbole (1992) discussed the self-normalized bounded law of iterated logarithm (SNBLIL) for B-valued random variables and gave the following definition for SNBLIL: $\{X_n, n \geqslant 1\}$ will be said to satisfy the SNBLIL ($\{X_n\} \in$ SNBLIL) if there exists a non-random constant $0 < M < \infty$ such that for some $p \in [1, 2]$ and $r > 0$,

$$\limsup_{n \to \infty} \frac{\|\sum_{i=1}^n X_i\|}{(\sum_{i=1}^n \|X_i\|^p)^{1/p} (L_2 \sum_{i=1}^n \|X_i\|^p)^r} = M \quad \text{a.s.} \tag{1.1}$$

Further, Godbole (1992) obtained the SNBLIL for $r = \dfrac{1}{2}$ and $r = (p-1)/p$.

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0167-7152/$ - see front matter
doi:10.1016/j.spl.2006.09.011

The advantage of SNBLIL is to drop the standard bounded assumption for the random variables $\{X_n, n \geqslant 1\}$. But these results in Godbole (1992) did not give the accurate value for the non-random constant M and deeply depend on the type of Banach spaces. In the case of real symmetric random variables, Marcinkiewicz proved that $M \leqslant 1$ for $p = 2$ and $r = \frac{1}{2}$ (see Griffin and Kuelbs, 1991 for a simple proof of Marcinkiewicz's result). Therefore, it was of interest to ask whether $M \leqslant 1$ for B-valued symmetric random variables or to give the accurate estimate for M. By replacing the self-normalizer $(\sum_{i=1}^{n} \|X_i\|^p)^{1/p}(L_2 \sum_{i=1}^{n} \|X_i\|^p)^{1/2}$ by $(\sup_{f \in B_1^*} \sum_{i=1}^{n} |f(X_i)|^p)^{1/p}(L_2 \sup_{f \in B_1^*} \sum_{i=1}^{n} |f(X_i)|^p)^{1/2}$, Deng (2003) answered this question and obtained the following theorems.

Theorem 1.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent symmetric B-valued random variables. Suppose that for some $p \in [1, 2]$, the following conditions hold*:

$$\lim_{n\to\infty} W_{n,p}^p \to +\infty \quad a.s. \tag{1.2}$$

$$\frac{\sup_{f \in D} |f(X_n)| (L_2 W_{n,p}^p)^{1/2}}{W_{n,p}} \to 0 \quad a.s. \tag{1.3}$$

$$S_n / (2 W_{n,p}^2 L_2 W_{n,p}^p)^{1/2} \to 0 \quad \text{in probability.} \tag{1.4}$$

Then

$$\limsup_{n\to\infty} \frac{\|S_n\|}{(2 W_{n,p}^2 L_2 W_{n,p}^p)^{1/2}} \leqslant 1 \quad a.s.$$

In particular, *if* (1.2), (1.3) *and* (1.4) *hold for $p = 2$*, *then*

$$\limsup_{n\to\infty} \frac{\|S_n\|}{(2 W_{n,2}^2 L_2 W_{n,2}^2)^{1/2}} = 1 \quad a.s.$$

Theorem 1.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent symmetric B-valued random variables. Suppose that for some $p \in [1, 2]$, the following conditions hold*:

$$\lim_{n\to\infty} W_{n,p}^p \to +\infty \quad a.s. \tag{1.5}$$

$$\frac{\sup_{f \in D} |f(X_n)| (L_2 W_{n,p}^p)^{(p-1)/p}}{W_{n,p}} \to 0 \quad a.s., \tag{1.6}$$

$$S_n / W_{n,p} (L_2 W_{n,p}^p)^{(p-1)/p} \to 0 \quad \text{in probability.} \tag{1.7}$$

Then

$$\limsup_{n\to\infty} \frac{\|S_n\|}{2 W_{n,p} (L_2 W_{n,p}^p)^{p-1/p}} \leqslant \begin{cases} \dfrac{1}{2} & p = 1, \\[2ex] \left(\dfrac{p}{p-1}\right)^{(p-1)/p} & 1 < p < 2 \ \ a.s., \\[2ex] \dfrac{\sqrt{2}}{2} & p = 2. \end{cases}$$

Note that the above theorems actually are the self-normalized versions of Kolmogorov's LIL. Therefore, it is of concern to ask whether the self-normalized version of Wittmann's LIL holds in Banach space. The main aim of the present paper is to solve the proposed questions. The paper is organized as follows. In Section 2, we state the main results. In Section 3, we develop a new inequality. The proofs of theorems are obtained by using an entropy approach and the new inequality.