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Algebraic polynomials with random non-symmetric coefficients

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Abstract

This paper provides an asymptotic formula for the expected number of zeros of a polynomial of the form $a_0(\omega) + a_1(\omega)\binom{n}{1}^{1/2}x + a_2(\omega)\binom{n}{2}^{1/2}x^2 + \cdots + a_n(\omega)\binom{n}{n}^{1/2}x^n$ for large n. The coefficients $\{a_j(\omega)\}_{j=0}^{n}$ are assumed to be a sequence of independent normally distributed random variables with fixed mean μ and variance one. It is shown that for μ non-zero this expected number is half of that for $\mu = 0$. This behavior is similar to that of classical random algebraic polynomials but differs from that of random trigonometric polynomials.

MSC: primary 60G99; secondary 60H99

1. Introduction

There has been much interest in the study of the behavior of random algebraic polynomials. These are defined as

$$Q_n(x) = \sum_{j=0}^{n} a_j(\omega)x^j, \tag{1.1}$$

where $\{a_j(\omega)\}_{j=0}^{n}$, $\omega \in \Omega$, is a sequence of independent random variables defined on a fixed probability space $(\mathcal{A}, \Omega, \Pr)$. Suppose, the $a_j(\omega)$ are identically distributed with $E(a_j(\omega)) = \mu$ and $\text{var}(a_j(\omega)) = 1$. Let $N_n(a, b)$ be the number of real zeros of $Q_n(x)$ in (a, b). For many classes of distributions it has been shown that for $\mu = 0$ the expected number of real roots, $EN_n(-\infty, \infty)$, is asymptotic to $(2/\pi)\log n$. For μ non-zero Ibragimov and Maslova (1971) show that this asymptotic value is reduced by half. Their results, based on Ibragimov and Maslova (1971), remain valid for a wide class of distributions of the coefficients. A recent work of Wilkins (1988), shows that in fact the error term in the above asymptotic formula is small.

For the random trigonometric polynomial,

$$T_n(\theta) = \sum_{j=0}^{n} a_j(\omega)\cos j\theta,$$

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the number of real roots behaves differently. From Dunnage (1966) we know for $\mu = 0$ that $EN_n(0, 2\pi) \sim 2n/\sqrt{3}$. This shows that they have significantly more zeros than algebraic polynomials. This number remains the same when we pass to the case of non-zero μ, see Farahmand (1987) or Sambandham and Renganathan (1981).

In this paper we consider random algebraic polynomials whose coefficients are independent but not identically distributed. Instead they can be written in the form

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j}^{1/2} a_j(\omega)x^j, \tag{1.2}$$

where the $a_j(\omega)$ have identical distributions. Physical applications of these polynomials can be found in Ramponi (1999). The mathematical behavior of $P_n(x)$ defined in (1.2) was presented for the first time by Edelman and Kostlan in their interesting work (Edelman and Kostlan, 1995), which includes many of the original approaches to the study of random algebraic polynomials. In the latter it is shown that $P_n(x)$ has significantly fewer zeros than trigonometric polynomials but more than algebraic ones. In particular for normally distributed coefficients with $\mu = 0$ we have $EN_n(-\infty, \infty) \sim \sqrt{n}$. Our interest is in the case of normally distributed coefficients with non-zero μ. The mathematical significance of non-zero mean coefficients is explained in Bharucha-Reid and Sambandham (1986) or Farahmand (1998, page 52). The latter includes a review of the recent developments of properties of $P_n(x)$, $Q_n(x)$, $T_n(x)$ and other related polynomials. In a similar direction the case of the polynomial $Q_n(x)$ with non-identically distributed coefficients is discussed in the recent work of Farahmand et al. (2002) and Farahmand and Nezakati (2005).

In the following theorem we show that, like the case of classical algebraic random polynomials $Q_n(x)$ and unlike that of random trigonometric polynomials $T_n(x)$, the asymptotic value of $EN_n(-\infty, \infty)$ for $P_n(x)$ for non-zero μ is half of that for $\mu = 0$. We prove:

Theorem 1. *If the coefficients $a_j(\omega)$ of $P_n(x)$ in (1.2) have identical normal distributions with $\mu \neq 0$ and unit variance, then for large n*

$$EN_n(-\infty, \infty) \sim \frac{\sqrt{n}}{2}.$$

2. Primary analysis

The Kac–Rice formula, Rice (1945) or Kac (1943), gives the expected number of real zeros of $P_n(x)$. We will use a generalization of this formula from Farahmand (1998, page 43). Using the notation from this source we put

$$A^2 = \mathrm{var}(P_n(x)) = \sum_{j=0}^{n} \binom{n}{j} x^{2j} = (x^2+1)^n \tag{2.1}$$

$$B^2 = \mathrm{var}(P_n'(x)) = \sum_{j=1}^{n} j^2 \binom{n}{j} x^{2j-2} = n(x^2+1)^{n-2}(nx^2+1) \tag{2.2}$$

$$C = \mathrm{cov}(P_n(x), \qquad P_n'(x)) = \sum_{j=1}^{n} j \binom{n}{j} x^{2j-1} = nx(x^2+1)^{n-1} \tag{2.3}$$

$$\alpha = E(P_n(x)) = \mu \sum_{j=0}^{n} \binom{n}{j}^{1/2} x^j \tag{2.4}$$

$$\beta = E(P_n'(x)) = \mu \sum_{j=1}^{n} j \binom{n}{j}^{1/2} x^{j-1} \tag{2.5}$$

we also need

$$\Delta^2 = A^2B^2 - C^2 = n(x^2+1)^{2n-2} \tag{2.6}$$