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Characterizations of multiparameter Cox and Poisson processes by the renewal property

Ely Merzbach*, Yair Y. Shaki

Department of Mathematics, Bar-Ilan University, 52900 Ramat Gan, Israel

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Abstract

In the first part of this paper, we give two characterizations of the multiparameter Poisson process: one characterization of the Poisson process uses the renewal property in the space $\mathbf{R}_+^d$, and the other characterization uses the exponential distributions of random areas of rectangles. Finally, we obtain a partial generalization of a characterization of the random measure associated with a renewal Cox process from $\mathbf{R}_+$ to $\mathbf{R}_+^d$.

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Keywords: Multi-parameter renewal process; Poisson process; Cox process

1. Introduction

In recent years, there have been many new results on the dynamical properties of random processes indexed by a multidimensional time parameter or by a class of sets. In particular, Ivanoff and Merzbach (2000) provide a definition of the renewal property for general point processes on $\mathbf{R}_+^d$ in a manner that includes the Poisson process.

In the first part of this paper, we give two characterizations of the spatial Poisson process: one characterization uses the renewal property in $\mathbf{R}_+^d$, and the other characterization uses the exponential distributions of random areas of rectangles. In the second part, we obtain a partial generalization of a characterization of the random measure associated with a renewal Cox process from $\mathbf{R}_+$ to $\mathbf{R}_+^d$.

2. Basic notation and definitions

The definitions and results in this section are taken from Ivanoff and Merzbach (2006). Points in $T = \mathbf{R}_+^d$ will be denoted by lower case letters such as s or t, and sets in T will be denoted by upper case letters. Families of sets in T will be denoted by script letters.

*Corresponding author. Tel.: +972 2 9941168; fax: +972 2 9947089.
E-mail addresses: merzbach@macs.biu.ac.il (E. Merzbach), yair_sh@shoresh.org.il (Y.Y. Shaki).

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$\mathscr{A}$ is the collection of rectangles $A_t := [0,t]$. For $t \in T$, E_t denotes the "future" of t: $E_t = \{s \in T : t \leqslant s\} = \{s \in T : A_t \subseteq A_s\}$. $\mathscr{A}(u)$ is the collection of finite unions of sets from $\mathscr{A}$. More generally, for any subset B of T, its past is defined to be $A(B) = \bigcup_{t \in B} A_t$. Finally, let $\mathscr{C}$ be the class of sets of the form $\mathscr{C} = A \backslash \bigcup_{i=1}^{k} A_i$ where $A, A_i \in \mathscr{A}$ and k is finite.

We assume the existence of a sufficiently rich probability space $(\Omega, \mathscr{F}, P)$ on which we define our processes (i.e., the probability space is assumed to be large enough so that each of the random elements defined subsequently is measurable). Our processes will be indexed by $\mathscr{A}$, and more generally, when an $\mathscr{A}$-indexed process induces a random measure on T, it may be parameterized by the collection $\mathscr{B}$ of Borel sets of T. Moreover, since a set in $\mathscr{A}$ is characterized by its upper right corner, we can identify any $\mathscr{A}$-indexed process X with its T-indexed counterpart $X_t = X_{[0,t]} = X_{A_t}$. For notational convenience, we will occasionally use $X(A)$ or $X(t)$ instead of X_A, respectively X_t.

Let $N = \{N_{A_t} = N_t; t \in T\}$ be a point process (i.e., an integer-valued random measure; cf. Daley and Vere-Jones, 1988). We will always assume that N is locally finite (i.e., $N_B < \infty; \forall B \in \mathscr{B}$ and B is compact) and that $N_t = 0$ if one or more of the coordinates of t is 0.

Definition 2.1. Let $N = \{N_{A_t} = N_t; t \in T\}$ be a point process on $T = \mathbf{R}_+^d$.

- N is *simple* if each realization of N satisfies $N_{\{t\}} = 0$ or 1 for all $t \in T$. (Note the distinction between $N_t = N_{A_t}$ and $N_{\{t\}}$, the mass of N on the singleton $\{t\}$.) If $N_{\{t\}} = 1$, then t is a *jump point* of N.
- N is *strictly simple* if whenever t is a jump point of a realization of N ($N_{\{t\}} = 1$), then $N(\partial A_t) = 1$ (i.e., there are no other jump points on ∂A_t).

Definition 2.2. For an arbitrary set $B \in \mathscr{B}$,

$$\min(B) = \{t \in B : s \nleqslant t, \forall s \in B \text{ such that } s \neq t\}.$$

The set B is called *incomparable* if $B = \min(B)$.

Definition 2.3. For $B \in \mathscr{B}$, we say that t is an *exposed point* of B if

- $t \in B$,
- $(E_t)^\circ \cap B = \emptyset$, and
- there exists $\varepsilon > 0$ such that for each coordinate $t^{(i)}$ of $t = (t^{(1)}, \ldots, t^{(d)})$, $A_{(t^{(1)}, \ldots t^{(i-1)}, t^{(i)} + \delta, t^{(i+1)}, \ldots, t^{(d)})} \subseteq B; \forall \delta \leqslant \varepsilon$.

The set of exposed points of B is denoted by $\varepsilon(B)$.

Definition 2.4. Let N be a point process.

- $\xi_n = \{t \in T : N_{t-} = N_{(0,t)} < n\}, n = 1, 2, \ldots$.
- $\xi_n^+ = \overline{(\bigcup_{k \neq j} (E_{\tau_k^{(n)}} \cap E_{\tau_j^{(n)}}))^c}$, where $\overline{(\cdot)}$ denote the closure operation. If ξ_n has only one exposed point, then $\xi_n^+ = T$.
- $\Delta_N := \{\tau : N_{\{\tau\}} = 1\}$ is the set of jump points of the process N.

Definition 2.5. A mapping $\xi : \Omega \longrightarrow \mathscr{A}(u)$ is called a random set if for every $t \in T$, $\{\omega : t \in \xi(\omega)\} \in \mathscr{F}$; i.e., ξ is a measurable mapping from $(\Omega, \mathscr{F})$ into $(\mathscr{A}(u), \mathscr{F}_{\mathscr{A}(u)})$, where $\mathscr{F}_{\mathscr{A}(u)} = \sigma\{\{D \in \mathscr{A}(u) : t \in D\}, t \in T\}$.

The class of sets of the form $\{D \in \mathscr{A}(u) : t_1, t_2, \ldots, t_n \in D\}$ for $t_1, t_2, \ldots, t_n \in T$ is a π-system generating $\mathscr{F}_{\mathscr{A}(u)}$, so that $P\xi^{-1}$ is determined by

$$P\xi^{-1}\{D : t_1, \ldots t_n \in D\} = P\{\omega : t_1, \ldots, t_n \in \xi(\omega)\}.$$

In other words, the distribution of a random set is characterized by these probabilities. We can define independence between random sets as usual (cf. Matheron, 1975; Stoyan et al., 1995): two random sets ξ and