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A note on the complete convergence of moving average processes

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ABSTRACT

Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of i.i.d. random variables with $E|Y_1| < \infty$, $\{a_i, -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. It is still an open question whether $\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{i=-\infty}^{\infty} \sum_{k=1}^{n} a_{i+k}(Y_i - EY_i)| > n\epsilon) < \infty$ for all $\epsilon > 0$. In this paper, we show that the answer to this question is false by giving a counterexample. This also shows that some basic results in the literature on this topic are not completely correct.

1. Introduction

Assume that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed random variables. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and

$$X_n = \sum_{i=-\infty}^{\infty} a_{i+n} Y_i, \quad n \geq 1$$

be the moving average process based on the sequence $\{Y_i\}$.

Under the independence assumption of the base sequence $\{Y_i\}$, many limiting results have been obtained for the moving average process $\{X_n, n \geq 1\}$. For example, Ibragimov (1962) has established the central limit theorem, Burton and Dehling (1990) have obtained a large deviation principle, and Li et al. (1992) have obtained the complete convergence. Under different dependence assumptions of the base sequence $\{Y_i\}$, Zhang (1996), Baek et al. (2003), and Li and Zhang (2004) have obtained the complete convergence results.

For a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables, Baum and Katz (1965) proved the following well-known complete convergence theorem.

Theorem A. *Suppose that $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables. Then $EX_1 = 0$ and $E|X_1|^{rp} < \infty (1 \leq p < 2$, $r \geq 1)$ if and only if $\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{n} X_i| > n^{1/p}\epsilon) < \infty$ for all $\epsilon > 0$.*

The case $r = 2$ and $p = 1$ of the above theorem was proved by Hsu and Robbins (1947) and Erdös (1949). Spitzer (1956) proved the above theorem for the case $r = 1$ and $p = 1$.

Li et al. (1992) generalized Hsu–Robbins–Erdös result for the moving average process based on a sequence of i.i.d. random variables $\{Y_i, -\infty < i < \infty\}$. Zhang (1996) and Baek et al. (2003) generalized the result of Baum and Katz (1965) for the moving average process based on a sequence of dependent random variables. If we omit the insignificant condition (slowly

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varying function), the result of Zhang (1996) can be formulated as follows:

Theorem B. *Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of identically distributed and ϕ-mixing random variables with $\sum_{n=1}^{\infty} \phi^{1/2}(n) < \infty$. Suppose that $\{X_n, n \geq 1\}$ is the moving average process based on the sequence $\{Y_i\}$. If $EY_1 = 0$ and $E|Y_1|^{rp} < \infty$ for some $1 \leq p < 2$ and $r \geq 1$, then*

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{k=1}^{n} X_k\right| > n^{1/p}\epsilon\right) < \infty \quad \text{for all } \epsilon > 0.$$

Baek et al. (2003) proved Theorem B for the negatively associated random variables. However, the proofs of Zhang (1996) and Baek et al. (2003) are mistakenly based on the fact that

$$\sum_{i=1}^{n} i^{r-1-1/p} = O(n^{r-1/p}). \tag{1}$$

Note that (1) holds only for $r - 1/p > 0$. From the conditions $1 \leq p < 2$ and $r \geq 1$, the proofs of Zhang (1996) and Baek et al. (2003) are valid except for the case $r = 1$ and $p = 1$. Thus it is natural to ask whether the result of Spitzer (1956) holds for the moving average process.

Question. If $\{Y_i, -\infty < i < \infty\}$ is a sequence of i.i.d. random variables with $E|Y_1| < \infty$, and $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers, then $\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{i=-\infty}^{\infty} \sum_{k=1}^{n} a_{i+k}(Y_i - EY_i)| > n\epsilon) < \infty$ for all $\epsilon > 0$?

In this paper, we show that the answer to the question is false by giving a counterexample. From this result, we have that Theorems of Zhang (1996) and Baek et al. (2003) for the case $r = 1$ and $p = 1$ are not true.

2. A counterexample

In this section, we give a counterexample to the question. To do this, we need the following lemmas. Lemma 1 is due to Etemadi (1985).

Lemma 1. *If $X_1, \ldots, X_n$ are independent random variables, then for any $t > 0$*

$$\max_{1 \leq l \leq n} P\left(\left|\sum_{i=1}^{l} X_i\right| > t\right) \geq \frac{1}{4} \sum_{i=1}^{n} P\left(|X_i| > 8t\right) \left\{1 - P\left(\max_{1 \leq l \leq n} \left|\sum_{i=1}^{l} X_i\right| > 4t\right)\right\}.$$

Lemma 2. *Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of i.i.d. non-negative random variables with $EY_1 < \infty$, $\{a_i, -\infty < i < \infty\}$ a summable sequence of non-negative real numbers. Then there exist positive constants C and D such that*

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} P\left(\sum_{k=1}^{n} a_{i+k} Y_i > \frac{n}{2}\right) \leq C \sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\sum_{i=-\infty}^{\infty} \sum_{k=1}^{n} a_{i+k}(Y_i - EY_i)\right| > \frac{n}{64}\right) + DEY_1.$$

Proof. Set $\theta^{-1} = \sum_{-\infty}^{\infty} a_i$ and $a_{ni} = \sum_{k=1}^{n} a_{i+k}$. Then it is obvious that $a_{ni} \leq 1/\theta$ and $\sum_{i=-\infty}^{\infty} a_{ni} \leq n/\theta$. By Lemma 1, we have that

$$\sum_{i=-\infty}^{\infty} P\left(a_{ni} Y_i > \frac{n}{2}\right) = \sum_{i=-\infty}^{\infty} P\left(a_{ni} Y_i I(Y_i > \theta n/2) > \frac{n}{2}\right) \leq \frac{4P\left(\sum_{i=-\infty}^{\infty} a_{ni} Y_i I(Y_i > \theta n/2) > \frac{n}{16}\right)}{1 - P\left(\sum_{i=-\infty}^{\infty} a_{ni} Y_i I(Y_i > \theta n/2) > \frac{n}{4}\right)}. \tag{2}$$

We also have by Markov's inequality that

$$\begin{aligned} P\left(\sum_{i=-\infty}^{\infty} a_{ni} Y_i I(Y_i > \theta n/2) > \frac{n}{4}\right) &\leq \frac{4}{n} \sum_{i=-\infty}^{\infty} a_{ni} E Y_1 I(Y_1 > \theta n/2) \\ &\leq \frac{4}{\theta} E Y_1 I(Y_1 > \theta n/2) \to 0 \end{aligned}$$

as $n \to \infty$. Hence there exists a positive integer N such that $\sum_{i=-\infty}^{\infty} a_{ni} EY_1 I(Y_1 > \theta n/2) \leq n/32$ and $P(\sum_{i=-\infty}^{\infty} a_{ni} Y_i I(Y_i > \theta n/2) > \frac{n}{4}) \leq 1/8$ if $n \geq N$. It follows that for $n \geq N$

$$1 - P\left(\sum_{i=-\infty}^{\infty} a_{ni} Y_i I(Y_i > \theta n/2) > \frac{n}{4}\right) \geq \frac{7}{8} \tag{3}$$