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On conditional maximum likelihood estimation for INGARCH(p, q) models

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ABSTRACT

We establish the strong consistency and asymptotic normality of the conditional maximum likelihood estimator (CMLE) for INGARCH(p, q) models. Moreover, we develop an efficient algorithm to compute the estimated information matrix of the CMLE so that statistical inferences are readily to be conducted.

1. Introduction

The integer-valued generalized autoregressive conditional heteroscedastic (INGARCH) models were originally developed by Ferland et al. (2006) in the same spirit of the GARCH models (Bollerslev, 1986). This family of models has attracted considerable attention in recent years due to its capability of modeling count time series. For example, the models can accommodate overdispersion often exhibited in count data and can allow for a variety of serial dependence structures (Weiß, 2009). According to Cox (1981), the INGARCH models are classified as "observation-driven" in the sense that serial dependence among the observations is introduced by making the conditional mean of the current observation depend explicitly upon the past observations.

The INGARCH(p, q) models have been studied by many researchers, including Weiß(2009), Fokianos et al. (2009), Fokianos and Fried (2010), Doukhan et al. (2012), and Kang and Lee (2014). However, most studies are confined to the INGARCH(1, 1) model and whence the results established are limited to the lower orders of $p = 1$ and $q = 1$. As to estimation of the model parameters, one of the commonly used methods is the conditional likelihood method. Fokianos et al. (2009) studied the conditional maximum likelihood estimator (CMLE) for the INGARCH(1, 1) model and showed its asymptotic normality via a perturbed model. Kang and Lee (2014), using a different approach, also established the asymptotic normality of the CMLE for the INGARCH(1, 1) model.

Following the strategy used by Kang and Lee (2014), in this paper we extend their asymptotic results to INGARCH(p, q) models in general. Since the INGARCH model bears similarities to the GARCH model, we adopt some key regularity conditions of Francq and Zakoïan (2010). Under these conditions, we can show that the CMLE is strongly consistent and asymptotically normal for any arbitrarily chosen initial values. The change-point tests of Kang and Lee (2014) are also extended to the general INGARCH(p, q) model. During the preparation of this manuscript, it came to our attention that in a recent paper

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Ahmad and Francq (2016) studied the Poisson quasi-maximum likelihood estimator of the conditional mean parameter of count time series. Since the INGARCH(p, q) model turns out to satisfy the conditions they impose, their asymptotic results apply. However, our derivations are essentially based on a different approach and somewhat different conditions.

Moreover, we develop an efficient algorithm to compute the estimated information matrix of the CMLE so that statistical inferences with respect to the parameters are readily to be conducted. To examine the finite-sample performance of the CMLE and the efficiency of the algorithm we conduct extensive simulation studies. The paper concludes with a real example. We study the counts of major hurricanes in the Atlantic basin between 1943 and 2014. Note that, there have been increased research interests in studying hurricane frequency and intensity since the Hurricane Katrina hit New Orleans, LA, in 2005. We fit INGARCH models of varied orders to the data and perform model diagnostics. A change-point test is applied to the fitted model. Our analysis supports the preference for a higher order INGARCH model rather than the widely-used INGARCH$(1, 1)$ model.

2. Conditional maximum likelihood estimation for INGARCH(p, q) model

Let $\{Y_t\}$ be a count time series and let $\mathcal{F}_t$ be the σ-field generated by $\{Y_s;\, s \le t\}$. The INGARCH(p, q) model is specified as

$$\begin{cases} Y_t|\mathcal{F}_{t-1} \sim \text{Poisson}(\lambda_t); \quad \forall\, t \in \mathbb{Z}, \\ \lambda_t = \delta + \sum_{i=1}^{p} \alpha_i \lambda_{t-i} + \sum_{j=1}^{q} \beta_j Y_{t-j}, \end{cases} \tag{1}$$

where $\{\lambda_t\} = E(Y_t|\mathcal{F}_{t-1})$ and $\delta > 0, \alpha_i \ge 0, i = 1, \ldots, p$, and $\beta_j \ge 0, j = 1, \ldots, q$ are model parameters.

2.1. Asymptotic results of CMLE

Let $\boldsymbol{\theta} = (\delta, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)^{\mathrm{T}}$ denote the vector of parameters, which belongs to the parameter domain $\Theta \subset (0, \infty) \times [0, \infty)^{p+q}$. Let $\boldsymbol{\theta}_0 = (\delta_0, \alpha_{01}, \ldots, \alpha_{0p}, \beta_{01}, \ldots, \beta_{0q})^{\mathrm{T}}$ denote the true value of $\boldsymbol{\theta}$. Suppose we have observed the data $Y_1, Y_2, \ldots, Y_n$. Then, if the initial value $\underline{\boldsymbol{\lambda}}_1 = (\lambda_1, \lambda_0, \ldots, \lambda_{2-p}; Y_0, Y_{-1}, \ldots, Y_{2-q})$ were known, the conditional likelihood would be $L(\boldsymbol{\theta}) = \prod_{t=1}^{n}[e^{-\lambda_t}\lambda_t^{Y_t}/(Y_t!)]$. In practice, an initial value $\hat{\underline{\boldsymbol{\lambda}}}_1 \ge \mathbf{0}$ is needed to obtain the sequential $\hat{\lambda}_t$'s via (1). As a result, the conditional likelihood function becomes $\prod_{t=1}^{n}[e^{-\hat{\lambda}_t}\hat{\lambda}_t^{Y_t}/(Y_t!)]$ and the corresponding log-likelihood function is $\hat{\mathcal{L}}_n(\boldsymbol{\theta}) = \sum_{t=1}^{n} \hat{\ell}_t(\boldsymbol{\theta}) = \sum_{t=1}^{n} \left\{ Y_t \log(\hat{\lambda}_t) - \hat{\lambda}_t - \log(Y_t!) \right\}$. The CMLE of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}_n$, is defined as the maximizer of $\hat{\mathcal{L}}_n(\boldsymbol{\theta})$; namely $\hat{\boldsymbol{\theta}}_n = \operatorname{argmax}_{\boldsymbol{\theta}\in\Theta} \hat{\mathcal{L}}_n(\boldsymbol{\theta})$. Define $\mathcal{A}_{\boldsymbol{\theta}}(z) = 1 - \sum_{i=1}^{p} \alpha_i z^i$ and $\mathcal{B}_{\boldsymbol{\theta}}(z) = \sum_{j=1}^{q} \beta_j z^j$; and by convention $\mathcal{A}_{\boldsymbol{\theta}}(z) \equiv 1$ if $p = 0$ and $\mathcal{B}_{\boldsymbol{\theta}}(z) \equiv 0$ if $q = 0$. Then, we can write the conditional mean process of (1) as $\mathcal{A}_{\boldsymbol{\theta}}(\mathrm{B})\lambda_t = \delta + \mathcal{B}_{\boldsymbol{\theta}}(\mathrm{B})Y_t$, where B is the back-shift operator. To derive the asymptotics of CMLE, we postulate the following regularity conditions:

(A1) $\sum_{i=1}^{p} \alpha_{0i} + \sum_{j=1}^{q} \beta_{0j} < 1$ and $\sum_{i=1}^{p} \alpha_i < 1$ for all $\boldsymbol{\theta} \in \Theta$.
(A2) When $p > 0$, $\mathcal{A}_{\boldsymbol{\theta}_0}(z)$ and $\mathcal{B}_{\boldsymbol{\theta}_0}(z)$ have no common roots, $\mathcal{B}_{\boldsymbol{\theta}_0}(1) \ne 0$, and $\alpha_{0p} + \beta_{0q} \ne 0$.
(A3) The parameter domain Θ is compact.
(A4) $\boldsymbol{\theta}_0$ is an interior point of Θ.

Condition (A1) ensures that the model possesses some key stability properties at $\boldsymbol{\theta}_0$. For example, Doukhan et al. (2012) have shown that under (A1) the INGARCH(p, q) model has a unique solution which is strictly stationary and ergodic (also c.f. Davis and Liu, in press). They also proved that under (A1) the INGARCH(p, q) process has finite moments of any order. Condition (A2) guarantees the identifiability of the higher order INGARCH(p, q) model. Under these conditions, the following statements hold true.

(B1) $\lambda_t(\boldsymbol{\theta}) = \lambda_t(\boldsymbol{\theta}_0)$ almost surely implies that $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.
(B2) For $\boldsymbol{\nu} \in \mathbb{R}^{p+q+1}$, $\boldsymbol{\nu}^{\mathrm{T}}[\partial\lambda_t/\partial\boldsymbol{\theta}] = 0$ almost surely implies that $\boldsymbol{\nu} = \mathbf{0}$.
(B3) There exists a constant $\lambda_L > 0$ such that $\inf_{\boldsymbol{\theta}\in\Theta} \lambda_t > \lambda_L$ and $E(\sup_{\boldsymbol{\theta}\in\Theta} \lambda_t) < \infty$ for all t.
(B4) There exist constants $K > 0$ and $\rho \in (0, 1)$ such that $\sup_{\boldsymbol{\theta}\in\Theta} |\omega_\ell^i(\boldsymbol{\theta})| \le K\rho^\ell$ and $\sup_{\boldsymbol{\theta}\in\Theta} |\omega_\ell^{i,j}(\boldsymbol{\theta})| \le K\rho^\ell$, where $\omega_\ell^i(\boldsymbol{\theta})$ and $\omega_\ell^{i,j}(\boldsymbol{\theta})$ are the terms, respectively, in

$$\frac{\partial\lambda_t}{\partial\theta_i} = \omega_0^i(\boldsymbol{\theta}) + \sum_{\ell=1}^{\infty} \omega_\ell^i(\boldsymbol{\theta})Y_{t-\ell} \quad \text{and} \quad \frac{\partial^2\lambda_t}{\partial\theta_i\theta_j} = \omega_0^{i,j}(\boldsymbol{\theta}) + \sum_{\ell=1}^{\infty} \omega_\ell^{i,j}(\boldsymbol{\theta})Y_{t-\ell}. \tag{2}$$

(B5) For any $\hat{\underline{\boldsymbol{\lambda}}}_1 \ge \mathbf{0}$, $\sup_{\boldsymbol{\theta}\in\Theta} |\hat{\lambda}_t(\boldsymbol{\theta}) - \lambda_t(\boldsymbol{\theta})| < \rho^t\xi$ for $1 \le t \le n$, where $\rho \in (0, 1)$ and ξ is an integrable random variable.

The asymptotic properties of the CMLE are stated as follows.

Theorem 1. *Suppose the conditions* (A1)–(A3) *are satisfied for the* INGARCH(p, q) *model* (1)*. For any arbitrarily chosen* $\hat{\underline{\boldsymbol{\lambda}}}_1 \ge \mathbf{0}$, $\hat{\boldsymbol{\theta}}_n$ *is strongly consistent. Moreover, if the condition* (A4) *is also satisfied, then* $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{D} \mathcal{N}\left(0, I^{-1}(\boldsymbol{\theta}_0)\right)$, *where* $I(\boldsymbol{\theta}_0) = E\left(\frac{\partial\ell_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\frac{\partial\ell_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}^{\mathrm{T}}}\right) = -E\left(\frac{\partial^2\ell_t(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^{\mathrm{T}}}\right)$.