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Constructing optimal asymmetric combined designs via Lee discrepancy

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ABSTRACT

The main objective of the present paper is to provide an answer to the question: How to construct optimal combined designs, whether regular or nonregular? In this paper, we take the Lee discrepancy as the optimality measure to construct the optimal mixed levels combined designs, which are most commonly used in practice. Equivalence between any combined design and its complementary combined design is investigated, which is a very useful constraint that reduce the search space for the optimal combined designs.

1. Introduction

How to obtain an effective design is a major concern of scientific research. Much effort has been spent on the measurement of design efficiency and the construction of effective designs. This topic always involves high-dimensional inputs with limited resources. The fractional factorial design (FFD) is one of the most popular methodologies to confrontation such difficulties. One consequence of using FFD's is the aliasing of factorial effects. Folding over FFD's is a method for breaking the links between aliased effects in a design. Adding to a design of odd resolution r a second fraction, then the combined design (CoD) has a resolution $r+1$. The new fraction, called the foldover design (FoD), is used to de-alias effects. In the last few years, much attention has been paid in employing the discrepancy to assess the optimal CoD(FoD). The reader can refer to Fang et al. (2003) and Elsawah and Qin (2015c, 2016b).

The main objective of the present paper is to provide an answer to the question: How to choose the FoD optimally, whether regular or nonregular? In this paper, we take the Lee discrepancy ($\mathcal{LD}$) as the optimality measure to assess the optimal FoD and take asymmetrical factorials with mixed two and three levels balanced designs (BD) as original designs. New and efficient analytical expressions and lower bounds of the $\mathcal{LD}$ are given for the CoD, which can be used as benchmarks for searching the optimal CoD. For any mixed two and three levels BD as the original design, CoD and its complementary CoD (CCoD) are defined. Equivalence between any CoD and its CCoD is investigated, which is a very useful constraint that reduce

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the search space for the optimal CoD. Illustrative examples are provided, where numerical studies lend further support to our theoretical results.

The remainder of this paper is organized as follows. Section 2 describes general structure of mixed two and three levels CoD based on $\mathcal{LD}$. In Section 3, new formulations of the $\mathcal{LD}$ for mixed two and three levels CoD are obtained. Equivalence between any CoD and its CCoD is investigated in Section 4. In Section 5, new and efficient lower bounds of the $\mathcal{LD}$ for symmetric and asymmetric levels CoD and BD are provided. Numerical examples are provided in Section 6.

2. Constructing asymmetric combined designs under $\mathcal{LD}$

To search an optimal design is an NP hard problem in the sense of computation complexity. Therefore, for reducing the computation complexity some structure of experimental points has to be considered. The so-called balanced levels designs have been widely used for construction of optimal designs. A balanced mixed levels design with m_1 two levels factors and m_2 three levels factors belonging to a class $\mathcal{B}(n; 2^{m_1} \times 3^{m_2})$ of designs is an $n \times m (m = m_1 + m_2)$ array $X = (x^1, \ldots, x^m)$ such that each of entries in each column x^i $(1 \le i \le m_1)$ takes values from a set of $\{0, 1\}$ equally often, and each of entries in x^i $(m_1 + 1 \le i \le m)$ takes values from a set of $\{0, 1, 2\}$ equally often. For any mixed two and three levels balanced design $\mathcal{H} \in \mathcal{B}(n; 2^{m_1} \times 3^{m_2})$, the Lee discrepancy (Zhou et al., 2008) measure of uniformity, denoted as $\mathcal{LD}(\mathcal{H})$, can be expressed in the following closed form

$$[\mathcal{LD}(\mathcal{H})]^2 = \mathcal{J} + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{m}\mathcal{R}_{ijk}, \tag{2.1}$$

where $\mathcal{J} = -\left(\frac{3}{4}\right)^{m_1}\left(\frac{7}{9}\right)^{m_2}$ and

$$\mathcal{R}_{ijk} = \begin{cases} 1 - \min\left\{\dfrac{|x_{ik} - x_{jk}|}{2}, 1 - \dfrac{|x_{ik} - x_{jk}|}{2}\right\} & \text{when } x_{ik} \in \{0, 1\}, \quad k = 1, \ldots, m_1, \\ 1 - \min\left\{\dfrac{|x_{ik} - x_{jk}|}{3}, 1 - \dfrac{|x_{ik} - x_{jk}|}{3}\right\} & \text{when } x_{ik} \in \{0, 1, 2\}, \quad k = m_1 + 1, \ldots, m. \end{cases}$$

Define $\Gamma = \{\mathfrak{T} : \mathfrak{T} = (\mathfrak{T}_1, \ldots, \mathfrak{T}_{m_1}, \mathfrak{T}_{m_1+1}, \ldots, \mathfrak{T}_m)\}$, where $\mathfrak{T}_i \in \{0, 1\}$ when $1 \le i \le m_1$ and $\mathfrak{T}_i \in \{0, 1, 2\}$ when $m_1 + 1 \le i \le m$, then for any $\mathfrak{T} \in \Gamma$, it defines a mixed levels foldover plan (FoP) for any balanced mixed levels design $\mathcal{H} \in \mathcal{B}(n; 2^{m_1} \times 3^{m_2})$, for $1 \le i \le m$, the ith factor is mapped to $x^i \uplus \mathfrak{T}_i$, where

$$x^i \uplus \mathfrak{T}_i = \begin{cases} (x_{1i} + \mathfrak{T}_i, \ldots, x_{ni} + \mathfrak{T}_i)' \pmod 2 & \text{when } 1 \le i \le m_1, \\ (x_{1i} + \mathfrak{T}_i, \ldots, x_{ni} + \mathfrak{T}_i)' \pmod 3 & \text{when } m_1 + 1 \le i \le m. \end{cases}$$

For any design $\mathcal{H} \in \mathcal{B}(n; 2^{m_1} \times 3^{m_2})$ and any FoP $\mathfrak{T} \in \Gamma$, the FoD, denoted by $\mathcal{F}$, is obtained by mapping the columns in $\mathcal{H}$ defined above according to the FoP $\mathfrak{T}$. Thus, it is to be noted that each FoD is generated by a FoP. The full design obtained by augmenting the runs of the FoD $\mathcal{F}$ to those of the original design $\mathcal{H}$ is called as CoD, denoted by $\mathcal{C}$, that is, $\mathcal{C} = \left(\mathcal{H}' \ \mathcal{F}'\right)'$. Denoted by $\mathbb{C}(n; 2^{m_1} \times 3^{m_2})$ the set of all mixed two and three levels CoD's.

Under any mixed levels FoP $\mathfrak{T} \in \Gamma$, the $\mathcal{LD}$ of the CoD $\mathcal{C}$, denoted by $[\mathcal{LD}(\mathcal{C})]^2$, can be calculated by the following formula.

Theorem 1. *For any mixed two and three levels CoD $\mathcal{C} \in \mathbb{C}(n; 2^{m_1} \times 3^{m_2})$, we have*

$$[\mathcal{LD}(\mathcal{C})]^2 = \mathcal{J} + \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{m}\mathcal{R}_{ijk} + \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{m}\mathcal{R}_{ijk}(\mathfrak{T}_k), \tag{2.2}$$

where

$$\mathcal{R}_{ijk}(\mathfrak{T}_k) = \begin{cases} 1 - \min\left\{\dfrac{|x_{ik} - x_{jk}(\mathfrak{T}_k)|}{2}, 1 - \dfrac{|x_{ik} - x_{jk}(\mathfrak{T}_k)|}{2}\right\} & \textit{when } x_{ik} \in \{0, 1\}, \quad k = 1, \ldots, m_1, \\ 1 - \min\left\{\dfrac{|x_{ik} - x_{jk}(\mathfrak{T}_k)|}{3}, 1 - \dfrac{|x_{ik} - x_{jk}(\mathfrak{T}_k)|}{3}\right\} & \textit{when } x_{ik} \in \{0, 1, 2\}, \quad k = m_1 + 1, \ldots, m \end{cases}$$

and

$$x_{jk}(\mathfrak{T}_k) = \begin{cases} (x_{jk} + \mathfrak{T}_k) \pmod 2 & \textit{when } x_{jk} \in \{0, 1\},\ k = 1, \ldots, m_1, \\ (x_{jk} + \mathfrak{T}_k) \pmod 3 & \textit{when } x_{jk} \in \{0, 1, 2\},\ k = m_1 + 1, \ldots, m. \end{cases}$$