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Moderate deviation principles for eigenvalues of β-Hermite and β-Laguerre ensembles with $\beta \to \infty$

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ABSTRACT

In this paper we consider the moderate deviation principles for the joint eigenvalues of β-Hermite and β-Laguerre ensembles when $\beta \to \infty$ with dimensions k fixed. Our proofs rely on finite dimensional matrix perturbation theory and some exponential equivalence estimates.

1. Introduction

Let A be a random $k \times k$ complex matrix while the real and imaginary parts are all independent and identically distributed standard normals and define $S = (A + A^{\star})/2$, where $A^{\star}$ is the conjugate transposed matrix of A. The matrix S is commonly known as the Gaussian Unitary Ensembles (GUE). The GUE and some other matrix models with dimensions $k \to \infty$ have been extensively studied, such as the weak convergence of empirical eigenvalue spectrums, almost surely convergence, large deviations of the extreme eigenvalues and so on. We will not give a detailed review about these facts here, one can refer to Anderson et al. (2010), Bai and Silverstein (2010) and Mehta (1991) as standard references about random matrix theory.

It is well known that the joint eigenvalues density function, $f_\beta^H(\lambda_1, \ldots, \lambda_k)$, of S (where $\beta = 2$) is given by (cf. Mehta, 1991)

$$f_\beta^H(\lambda_1, \ldots, \lambda_k) = c_{n,\beta} \prod_{1 \le i < j \le k} |\lambda_i - \lambda_j|^\beta e^{-\sum_{i=1}^k \lambda_i^2/2} := c_{n,\beta} |\Delta(\lambda_1, \ldots, \lambda_k)|^\beta e^{-\sum_{i=1}^k \lambda_i^2/2}, \tag{1}$$

where $c_{n,\beta}$ is the normalization constant. For general β, we refer above model as β-Hermite ensembles. The parameter β is interpreted as the inverse temperature in literature. As β goes to 0, the strength of the repulsion $|\Delta(\lambda_1, \ldots, \lambda_k)|^\beta$ decreases to 1, λ_i's would reduce to independent Gaussians and the randomness increases. In the frozen state, i.e., $\beta = \infty$, we can imagine the k eigenvalues fixed at the roots of the Hermite polynomial.

Dumitriu and Edelman (2005) establish the law of large number and central limit theorem for $(\lambda_1, \ldots, \lambda_k)$ as $\beta \to \infty$. Their proofs rely on a genius tridiagonal matrix representation of above models, see Section 3 for details. Using this approach,

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they prove the fluctuation limit distribution is Gaussian, which is quite different from the case $k \to \infty$ with β fixed, where the fluctuation limit is Tracy Widom law (cf. Johansson, 1998, Johansson, 2000, Johnstone, 2001, Tracy and Widom, 1996, Tracy and Widom, 2000). Dumitriu and Edelman's result shows that the empirical spectrums distribution can be approximated by Gaussian distribution, which also differs from $k \to \infty$ with β fixed (for example, $\beta = 2$, the empirical spectrums distribution has a limit known as semicircle law).

In this paper, we mainly focus on the case in which the dimension k is fixed and β is a parameter. A moderate deviation principle (MDP) of $(\lambda_1, \ldots, \lambda_k)$ as $\beta \to \infty$ will be established. The MDP characterizes the convergence speed of $(\lambda_1, \ldots, \lambda_k)$ towards to its fluctuation limit. As pointed by Gao and Zhao (2011), if the MDP holds for some statistics and we use the statistics to construct rejection(or acceptance) region, then the two types error probabilities tend to zero with exponentially rate. Thus, from the viewpoint of statistical costs of experiments, the moderate deviation estimation is quite meaningful. More applications of MDP can be found in Gao and Zhao (2011).

Besides the β-Hermite model considered in (1), we also consider the $k \times k$ Laguerre ensembles (which is also known as the Wishart matrix for $\beta = 2$). Its joint eigenvalues density function has form of

$$f^L_{\beta,a}(\lambda_1, \ldots, \lambda_k) = c^L_{\beta,a} \prod_{1\le i<j\le k} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{k} \lambda_i^{a-(k-1)\beta/2-1} e^{-\sum_{i=1}^{k} \lambda_i/2}, \tag{2}$$

where $\beta > 0$, parameter $a > (k-1)\beta/2$ and $c^L_{\beta,a}$ is the normalization constant. A similar MDP result is also presented in Section 4.

To end the introduction, we give some definitions in large deviation theory, a standard reference is Dembo and Zeitouni (2009). Let $(\mathcal{S}, d)$ be a metric space and $\{Y_n : n \ge 1\}$ be a sequence of $\mathcal{S}$-valued random variables on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\lambda(n)$ be a sequence of positive real numbers satisfying $\lambda(n) \to \infty$ as $n \to \infty$. A function $I(\cdot) : \mathcal{S} \to [0, +\infty]$ is said to be a rate function if it is lower semicontinuous. It is said to be a good rate function if its level set $\{x \in \mathcal{S} : I(x) \le l\}$ is compact for all $l \ge 0$. The sequence $\{Y_n, n \ge 1\}$ is said to satisfy a large deviation principle (LDP) with speed $\lambda(n)$ and with good rate function I if for any Borel set Γ in $\mathcal{S}$

$$\begin{aligned} -\inf_{x\in\Gamma^\circ} I(x) &\le \liminf_{n\to\infty} \frac{1}{\lambda(n)} \log \mathbb{P}(Y_n \in \Gamma) \\ &\le \limsup_{n\to\infty} \frac{1}{\lambda(n)} \log \mathbb{P}(Y_n \in \Gamma) \le -\inf_{x\in\overline{\Gamma}} I(x), \end{aligned}$$

where Γ° and $\overline{\Gamma}$ are the interior and closure of Γ, respectively.

For a family of random variables $\{\xi_n, n \ge 1\}$, assume that it satisfies a fluctuation theorem such as central limit theorem, if there exists a sequence $b(n)$ such that $b(n)(\xi_n - \theta) \to Y$ in law, where θ is a constant and Y is a nontrivial random variable. Generally, a LDP for $\{Y_n := r(n)(\xi_n - \theta), n \ge 1\}$ is called a MDP for $\{\xi_n, n \ge 1\}$, where $r(n)$ is an intermediate scale between 1 and $b(n)$, that is, $r(n) \to \infty$ and $b(n)/r(n) \to \infty$.

The paper is organized as follows. In Section 2, we give two preliminary lemmas which are crucial for proofs. The MDPs for $(\lambda_1, \ldots, \lambda_k)$ in β-Hermite and β-Laguerre ensembles are studied in the subsequent two sections separately.

2. Some preliminary lemmas

In this section we present two lemmas which we need in the proofs of our main results. For the $k \times k$ matrix $A = (a_{ij})$, let $\|A\|_{HS} := \sqrt{\sum_{i,j=1}^k a_{ij}^2}$ denotes the Hilbert–Smith norm of A. The first lemma involves the perturbation theory about eigenvalues of finite dimensional matrix. It is a simplified version of Lemma 6 from Chen et al. (2014). The proof based on Gershgorin disks theorem and matrix perturbation theories, one can refer to references therein for more details.

Lemma 2.1. *Let A and B be $k \times k$ symmetric matrices. Assume A has distinct eigenvalues and with unit eigenvector matrix Q. Set $\delta = 1 \wedge \min_{2\le i\le k}(\lambda_i(A) - \lambda_{i-1}(A))$, where $\lambda_i(X)$ denote the ith eigenvalue of X. Assume $M = A + \epsilon B$ with $\varepsilon > 0$, satisfying*

$$0 < \varepsilon \le \frac{\delta}{16k(1 + \|A\|_{HS})}.$$

Then we have

$$\max_{1\le i\le k} \left|\lambda_i(M) - \lambda_i(A) - \varepsilon q_i^T B q_i\right| \le \frac{32k\varepsilon^2 \|B\|_{HS}^2}{\delta},$$

where q_i is the ith column of Q.