Statistics and Probability Letters 118 (2016) 60–69

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On Kummer's distribution of type two and a generalized beta distribution

Marwa Hamza, Pierre Vallois *

Université de Lorraine, Institut de Mathématiques Elie Cartan, INRIA-BIGS, CNRS UMR 7502, BP 239, F-54506 Vandœuvre-lès-Nancy Cedex, France

ARTICLE INFO

Article history:
Received 11 September 2015
Received in revised form 14 March 2016
Accepted 15 March 2016
Available online 24 June 2016

MSC:
60B10
60F15
60G40
60G50
60J10

Keywords:
Generalized beta distribution
Kummer distribution of type two
Letac Wesołowski Matsumoto
Yor independence property

ABSTRACT

We characterize the Kummer distributions of type two (resp. the generalized beta distributions) as solution of an equation involving gamma (resp. beta) distributions. We give also some almost sure realizations of Kummer's distributions and generalized beta ones using the conditioning method and the rejection method as an application.

1. Introduction

(1) According to Koudou and Vallois (2012), a bijective and decreasing function f from $]0, +\infty[$ to $]0, +\infty[$ is said to be a Letac Wesołowski Matsumoto Yor (LWMY) function if there exist two positive and independent random variables X and Y such that $f(X+Y)$ and $f(X)-f(X+Y)$ are independent. It is has been proved in Letac and Wesołowski (2000) and Matsumoto and Yor (2001) that the function $f_0(x) = 1/x$ is a LWMY function and the related distributions are Generalized Inverse Gaussian and Gamma. Under additional assumptions, Koudou and Vallois (2012) have shown that there exist 4 classes of LWMY functions including the one generated by f_0. The three other classes are generated respectively by $f_1(x) = \frac{1}{e^x-1}$, $g_1(x) = f_1^{-1}(x)$ and $f_\delta^*(x) = \ln\left(\frac{e^x+\delta-1}{e^x-1}\right)$ where $x > 0$ and $\delta > 0$.

(2) The two cases $f = f_1$ and $f = g_1$ are "equivalent" and lead to Gamma and Kummer's of type two distributions. Recall that: $\gamma(\lambda, c)(dx) = \frac{c^\lambda}{\Gamma(\lambda)} x^{\lambda-1}e^{-cx}\, \mathbf{1}_{]0,+\infty[}(x)dx$ and $\beta(a, b)(dx) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\, \mathbf{1}_{[0,1]}(x)dx$, where $a, b, c, \lambda > 0$. As usual, $\gamma(\lambda)$ stands for $\gamma(\lambda, 1)$. The density function of the Kummer distribution of type two, with parameters $a, c > 0$ and $b \in \mathbb{R}$ is:

$$K^{(2)}(a, b, c)(dx) = \frac{1}{\Gamma(a)\, \Psi(a, 1-b, c)} x^{a-1}(1+x)^{-a-b} e^{-cx}\, \mathbf{1}_{]0,+\infty[}(x)dx \tag{1.1}$$

* Corresponding author.
E-mail address: pierre.vallois@univ-lorraine.fr (P. Vallois).

http://dx.doi.org/10.1016/j.spl.2016.03.014

where Ψ is the hyper-geometric confluent function of type 2 defined for $a, c > 0$ and $b \in \mathbb{R}$ by $\Psi(a,b,c) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1+t)^{b-a-1} e^{-ct}\, dt$.

The distributions $K^{(2)}(a,b,c)$ are the members of the natural exponential families generated by the measures $x^{a-1}(1+x)^{-a-b} \mathbf{1}_{]0,+\infty[}(x)\, dx$, for more details on the natural exponential families, see for instance Letac (1992).

Below, $L(X)$ stands for the law of the random variable X. Let us recall a weak version of Theorem 2.1 in Koudou and Vallois (2011).

Theorem 1.1. *Let X and Y be two independent and positive random variables such that:*

$$L(X) = K^{(2)}(a,b,c) \quad and \quad L(Y) = \gamma(b,c), \tag{1.2}$$

where $a, b, c > 0$. Then, the random variables:

$$U := \frac{1+X+Y}{1+X}\frac{X}{X+Y}, \quad V := X+Y \tag{1.3}$$

are independent and

$$L(U) = \beta(a,b) \quad and \quad L(V) = K^{(2)}(a+b,-b,c). \tag{1.4}$$

Theorem 2.1 in Koudou and Vallois (2011) says more generally that if X and Y are independent r.v.'s whose log densities are locally integrable, then U and V defined by (1.3) are independent if and only if (1.2) holds.

Let X and Y be two positive and independent random variables, Wesołowski (2015) has given a weaker condition to prove that U and V defined by (1.3) are independent if and only if (1.2) holds. Koudou (2012) has extended Theorem 1.1 to matrix variate distributions.

Note that Theorem 1.1 implies the following relation of convolution

$$K^{(2)}(a,b,c) * \gamma(b,c) = K^{(2)}(a+b,-b,c). \tag{1.5}$$

(3) Concerning the last class of LMWY functions which is generated by the function f_δ^*, with a suitable change of variable Koudou and Vallois (2012) have shown that the Matsumoto–Yor independence property takes the following form: we look for two independent random variables X and Y such that $f_\delta(XY)$ and $\frac{f_\delta(X)}{f_\delta(XY)}$ are independent, where:

$$f_\delta(x) = \frac{1-x}{1+(\delta-1)x}, \quad 0 < x < 1. \tag{1.6}$$

Under additional assumptions, see Theorem 3.3, the law of X (resp. Y) belongs to the family of generalized beta distributions $\{\beta_\delta(a,b,c), a,b,\delta > 0, c \in \mathbb{R}\}$ (resp. beta distributions), where

$$\beta_\delta\,(a,b,c)(dx) = N_\delta(a,b,c)\, x^{a-1}\,(1-x)^{b-1}\,(1+(\delta-1)x)^c\, \mathbf{1}_{[0,1]}(x)\, dx, \quad a,b,\delta > 0,\ c \in \mathbb{R} \tag{1.7}$$

and $N_\delta(a,b,c)$ is the normalization constant. Note that $\beta_\delta\,(a,b,c) = H(-c, a; a+b; 1-\delta)$, where $H(a,b;c;z)$ stands for the hyper-geometric distribution as defined in Chamayou and Letac (1991). Armero and Bayarri (1994) have considered the Gauss hypergeometric distribution $GH(\alpha,\beta,\gamma,z) := \beta_{1+z}\,(\alpha,\beta,-\gamma)$ as a marginal prior distribution. However, in this paper, we conserve the notations of Koudou and Vallois (2011). More recently, Ristić et al. (2015) have introduced "skewed" distributions associated with $\beta_\delta\,(a,b,c)$ and baseline cumulative distribution function F as cdf of the type: $F_{LN}(x) := F_\delta\,(a,b,c)\big(F(x)\big)$ where $F_\delta\,(a,b,c)$ is the cdf of $\beta_\delta\,(a,b,c)$. The authors have studied the moments, the Laplace transforms of such distributions. In the particular case where F is an exponential cdf, i.e. $F(x) = 1 - e^{-\lambda x}$, a maximum-likelihood estimation of parameters is given and the flexibility of these family of distributions is emphasized with real data sets.

(4) It is clear that if $b \neq 0$, then $K^{(2)}(a,b,c)$ is not equal to $K^{(2)}(a+b,-b,c)$. Therefore, either relation (1.5) or transformation $(x,y) \mapsto \left(\frac{1+x+y}{1+x}\frac{x}{x+y},\ x+y\right)$ considered in Theorem 1.1 do not give a closed identity satisfied by the Kummer distribution $K^{(2)}(a,b,c)$. We will consider in Proposition 2.1 a new transformation which permits to get a characterization of Kummer's distributions involving the gamma ones:

Theorem 1.2. *Let X, Y_1 and Y_2 be independent and positive random variables such that $L(Y_1) = \gamma(a,c)$ and $L(Y_2) = \gamma(a+b,c)$, $a, c > 0$, $b > -a$.*

1. *Then*

$$L(X) = L\left(\frac{Y_1}{1+\frac{Y_2}{1+X}}\right) \quad if\ and\ only\ if \quad L(X) = K^{(2)}(a,b,c). \tag{1.8}$$