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Strong law of large numbers for continuous random dynamical systems

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ABSTRACT

We study a dynamical system generalizing continuous iterated function systems and stochastic differential equations disturbed by Poisson noise. The aim of this paper is to study stochastic processes whose paths follow deterministic dynamics between random times, jump times, at which they change their position randomly. Continuous random dynamical systems can be used as a description of many physical and biological phenomena. We prove the existence of an exponentially attractive invariant measure and the strong law of large numbers for continuous random dynamical systems. We illustrate the usefulness of our criteria for asymptotic stability by considering a general d-dimensional model for the intracellular biochemistry of a generic cell with a probabilistic division hypothesis (see Lasota and Mackey, 1999).

1. Introduction

In the present paper we are concerned with the problem of proving the existence of an exponentially attractive invariant measure and the law of large numbers (LLN) for continuous random dynamical systems.

Continuous random dynamical systems (Horbacz, 2013) take into consideration some very important and widely studied cases, namely dynamical systems generated by learning systems (Barnsley et al., 1988; Iosifescu and Theodorescu, 1969; Karlin, 1953; Lasota and Yorke, 1994), random dynamical systems (Horbacz, 2008), continuous iterated function systems (Horbacz and Szarek, 2001), iterated function systems with an infinite family of transformations (Lasota and Mackey, 1999; Tyrcha, 1988; Tyson and Hannsgen, 1988), Poisson driven stochastic differential equations (Horbacz, 2006; Lasota and Traple, 2003), random evolutions (Griego and Hersh, 1969; Pinsky, 1991) and irreducible Markov systems (Werner, 2005). So called irreducible Markov systems introduced by Werner are used for the computer modelling of different stochastic processes.

A large class of applications of such models, both in physics and biology, is worth mentioning here: the growth of the size of structural populations, the motion of relativistic particles, both fermions and bosons (see Frisch, 1986, Keller, 1964), the generalized stochastic process introduced in the recent model of gene expression by Lipniacki et al. (2006).

On the other hand, it should be noted that most Markov chains may be represented by continuous iterated function systems. This turned out to be a very useful tool in the theory of cell cycles, for example: in a general d-dimensional model

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for the intracellular biochemistry of a generic cell with a probabilistic division hypothesis (see Lasota and Mackey, 1999). See also Tyson and Hannsgen (1988) or Murray and Hunt (1993) to get more details on the subject. Lasota and Mackey proved only stability, while we managed to evaluate rate of convergence, bringing some information important from biological point of view. This stability means that the cell division process leads to a population, which is unique with respect to cell function and structure, independently of the way in which the cells are prepared in the beginning. Hille et al. (in press) proposed the generalization of the model of the cell division and assumed the existence of a unique invariant measure in it.

The aim of this paper is to study stochastic processes whose paths follow deterministic dynamics between random times, jump times, at which they change their position randomly. Hence, we analyse stochastic processes in which randomness appears at times $t_0 < t_1 < t_2 < \cdots$. We assume that a point $x_0 \in Y$ moves according to one of the dynamical systems $S_i : \mathbb{R}_+ \times Y \to Y$ from some set $\{S_1, \dots, S_N\}$. The motion of the process is governed by the equation $X(t) = S_i(t, x_0)$ until the first jump time t_1. Then, we choose a transformation $q : Y \times \Theta \to Y$ from a family $\{q(\cdot, s) : s \in \Theta = [0, T]\}$ and define $x_1 = q(S_i(t_1, x_0), s)$. The process restarts from that new point x_1 and continues as before. This gives the stochastic process $\{X(t)\}_{t\geq 0}$ with jump times $\{t_1, t_2, \dots\}$ and post jump positions $\{x_1, x_2, \dots\}$. The probability determining the frequency with which the dynamical systems S_i are chosen is described by a matrix of probabilities $\pi_{ij} : Y \to [0, 1]$. The maps $q(\cdot, s)$ are randomly chosen with place dependent absolutely continuous distribution.

We are interested in the evolution of distributions of these random dynamical systems. We formulate criteria for the existence of an exponentially attractive invariant measure and the strong law of large numbers for such systems. Our results are based on an exponential convergence theorem due to Ślęczka and Kapica (see Kapica and Ślęczka, submitted for publication) and a version of the law of large numbers due to Shirikyan (see Shirikyan, 2003).

The results of this paper are related to previously published papers (Horbacz, 2013; Horbacz and Ślęczka, 2016; Wojewódka, 2013). The simplest case when Θ is equal to the finite set $\{1, \dots, K\}$ and $q_s : s \in \Theta$ are randomly chosen with discrete distribution is considered in Horbacz and Ślęczka (2016). In Horbacz (2013) we formulate criteria only for stability for continuous random dynamical systems. Exponential rate of convergence for continuous iterated function systems is considered in Wojewódka (2013). Our result generalized Theorem 2 from Wojewódka (2013). Additionally, the assumption (V) of Theorem 2, Wojewódka (2013) is restrictive if we want to apply our model in biology (see Powell, 1958).

The law of large numbers, which we study in this note, was also considered in many papers. Komorowski et al. (2010) obtained the weak law of large numbers for the passive tracer model in a compressible environment and Walczuk studied Markov processes with the transfer operator having spectral gap in the Wasserstein metric and proved the LLN in the non-stationary case (Walczuk, 2008).

2. Notation and basic definitions

Let (X, d) be a *Polish* space, i.e. a complete and separable metric space and denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X. By $B(X)$ we denote the space of bounded Borel-measurable functions equipped with the supremum norm, $C(X)$ stands for the subspace of bounded continuous functions. Let $\mathcal{M}(X)$ and $\mathcal{M}_1(X)$ be the sets of Borel measures on X such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}(X)$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1(X)$. The elements of $\mathcal{M}_1(X)$ are called *probability measures*. The elements of $\mathcal{M}(X)$ for which $\mu(X) \leq 1$ are called *subprobability measures*. By supp μ we denote the support of the measure μ. We also define

$$\mathcal{M}_1^1(X) = \left\{ \mu \in \mathcal{M}_1(X) : \int_X d(x, x_*)\mu(dx) < \infty \right\},$$

where $x_* \in X$ is fixed. By the triangle inequality this family is independent of the choice of x_*.

To simplify notation, we write

$$\langle f, \mu \rangle = \int_X f(x)\mu(dx) \quad \text{for} f \in B(X), \mu \in \mathcal{M}(X).$$

The space $\mathcal{M}_1(X)$ is equipped with the *Fourtet–Mourier metric*:

$$\|\mu_1 - \mu_2\|_{FM} = \sup\{|\langle f, \mu_1 - \mu_2 \rangle| : f \in \mathcal{F}\},$$

where $\mathcal{F} = \{f \in C(X) : |f(x) - f(y)| \leq d(x, y) \text{ and } |f(x)| \leq 1 \text{ for } x, y \in X\}$. An operator $P : \mathcal{M}(X) \to \mathcal{M}(X)$ is called a *Markov operator* if

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad \text{for} \lambda_1, \lambda_2 \geq 0, \ \mu_1, \mu_2 \in \mathcal{M}(X),$$
$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}(X).$$

A Markov operator P for which there exists a linear operator $U : B(X) \to B(X)$ such that

$$\langle Uf, \mu \rangle = \langle f, P\mu \rangle \quad \text{for } f \in B(X), \mu \in \mathcal{M}(X)$$

is called a *regular operator*. Furthermore, a regular Markov operator is Feller if $U(C(X)) \subset C(X)$.

We say that $\mu_* \in \mathcal{M}_1(X)$ is *invariant* for P if $P\mu_* = \mu_*$. An invariant measure μ_* is *attractive* if

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\|_{FM} = 0 \quad \text{for } \mu \in \mathcal{M}_1(X).$$