Statistics and Probability Letters 118 (2016) 80–86

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A strong law of large numbers for nonnegative random variables and applications

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ARTICLE INFO

Article history:
Received 4 February 2016
Received in revised form 18 June 2016
Accepted 20 June 2016
Available online 28 June 2016

MSC:
60F15

Keywords:
Strong law of large numbers
Weighted sums
Dependent random variables

ABSTRACT

For a sequence of nonnegative random variables $\{X_n, n \geq 1\}$ with finite means and partial sums $S_n = \sum_{i=1}^n X_i, n \geq 1$, and a sequence of positive numbers $\{b_n, n \geq 1\}$ with $b_n \uparrow \infty$, sufficient conditions are given under which $(S_n - ES_n)/b_n \to 0$ almost surely. Our result generalizes the strong law of large numbers obtained by Korchevsky (2015). Some applications for dependent random variables are also provided.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Set $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. We say that the sequence $\{X_n, n \geq 1\}$ satisfies the strong law of large numbers (SLLN) if

$$\frac{S_n - ES_n}{n} \to 0 \text{ almost surely (a.s.).} \tag{1.1}$$

There are many SLLNs for independent random variables. Kolmogorov proved a SLLN under the condition

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}(X_n)}{n^2} < \infty. \tag{1.2}$$

Petrov (1969) obtained a SLLN under the condition

$$\mathrm{Var}(S_n) = O\left(\frac{n^2}{\psi(n)}\right) \text{ for some } \psi \in \Psi_c, \tag{1.3}$$

where Ψ_c is the set of all functions $\psi(x)$ such that $\psi(x)$ is positive and nondecreasing in (x_0, ∞) for some $x_0 > 0$ and

$$\sum_{n=1}^{\infty} \frac{1}{n\psi(n)} < \infty. \tag{1.4}$$

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http://dx.doi.org/10.1016/j.spl.2016.06.017

The condition (1.3) is called Petrov's condition. Since $\psi(x)$ is positive and nondecreasing, (1.4) is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{\psi(\alpha^n)} < \infty \quad \text{for any } \alpha > 1. \tag{1.5}$$

Since $\{X_n, n \geq 1\}$ are independent, $\mathrm{Var}(S_n) = \sum_{i=1}^n \mathrm{Var}(X_i)$ and so it is easy to show that (1.3) implies (1.2).

Now we consider SLLNs for sequences of nonnegative random variables without independence. Etemadi (1983a) proved in an elementary way the following SLLN for nonnegative random variables.

Theorem 1.1 (*Etemadi, 1983a*)**.** *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables with finite second moments. Suppose that the following conditions hold:*

(i) $\sup_{i\geq 1} EX_i < \infty$,
(ii) $EX_iX_j \leq EX_iEX_j$ *for all* $j > i$,
(iii) $\sum_{i=1}^{\infty} \mathrm{Var}(X_i)/i^2 < \infty$.

Then $(S_n - ES_n)/n \to 0$ *a.s.*

Etemadi (1983b) also obtained a SLLN for weighted sums of nonnegative random variables.

Theorem 1.2 (*Etemadi, 1983b*)**.** *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables with finite second moments. Let $\{w_n, n \geq 1\}$ be a sequence of positive numbers with $w_n/W_n \to 0$ and $W_n \to \infty$, where $W_n = \sum_{i=1}^n w_i$ for $n \geq 1$. Suppose that the following conditions hold:*

(i) $\sup_{i\geq 1} EX_i < \infty$,
(ii) $\sum_{j=1}^{\infty} \sum_{i=1}^{j} w_i w_j \mathrm{Cov}^+(X_i, X_j)/W_j^2 < \infty$.

Then $\sum_{i=1}^n w_i(X_i - EX_i)/W_n \to 0$ *a.s.*

It is easy to show that Theorem 1.2 embraces Theorem 1.1 by letting $w_n = 1$ for $n \geq 1$. Extending the method of Etemadi (1983a), Csörgő et al. (1983) proved an analogue of Kolmogorov's SLLN for pairwise independent random variables. We can write the result of Csörgő et al. (1983) as a SLLN for nonnegative random variables by observing their proof carefully.

Theorem 1.3 (*Csörgő et al., 1983*)**.** *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables with finite second moments. Suppose that the following conditions hold:*

(i) $\sup_{n\geq 1} ES_n/n < \infty$,
(ii) $EX_iX_j \leq EX_iEX_j$ *for all* $j > i$,
(iii) $\sum_{n=1}^{\infty} \mathrm{Var}(X_n)/n^2 < \infty$.

Then $(S_n - ES_n)/n \to 0$ *a.s.*

The above theorem shows that the uniform boundedness of $\{EX_n, n \geq 1\}$ in Theorem 1.1 can be replaced by a weaker condition of Cesàro uniform boundedness (i.e., $\sup_{n\geq 1} \sum_{i=1}^n EX_i/n < \infty$). Chandra and Goswami (1992) proved a SLLN from the arguments of Csörgő et al. (1983).

Theorem 1.4 (*Chandra and Goswami, 1992*)**.** *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables with finite second moments. Let $\{b_n, n \geq 1\}$ be a nondecreasing unbounded sequence of positive numbers. Suppose that the following conditions hold:*

(i) $\sup_{n\geq 1} ES_n/b_n < \infty$,
(ii) *there exists a double sequence* $\{\rho_{ij}\}$ *of nonnegative reals such that*

$$\mathrm{Var}(S_n) \leq \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} \quad \textit{for all } n \geq 1,$$

(iii) $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij}/b^2_{\max\{i,j\}} < \infty$.

Then $(S_n - ES_n)/b_n \to 0$ *a.s.*

Setting $b_n = n$ for $n \geq 1$, $\rho_{ii} = \mathrm{Var}(X_i)$ for $i \geq 1$, and $\rho_{ij} = 0$ for $i \neq j$, Theorem 1.3 follows directly from Theorem 1.4.

Some SLLNs for nonnegative random variables have been established under Petrov's conditions. Petrov (2009) obtained the following SLLN for nonnegative random variables satisfying some moment conditions.

Theorem 1.5 (*Petrov, 2009*)**.** *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables satisfying Petrov's condition (1.3) and*

$$E(S_n - S_m) \leq C(n - m) \quad \textit{for sufficiently large } n - m, \tag{1.6}$$

where C is a positive constant. Then $(S_n - ES_n)/n \to 0$ a.s.