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On the strong laws of large numbers for weighted sums of random variables

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ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of real numbers with $0 < b_n \uparrow \infty$. Sufficient conditions are given under which $\sum_{i=1}^{n} a_i X_i / b_n \to 0$ almost surely. No conditions are imposed on the joint distributions of the $\{X_n\}$. Our results generalize and improve some known results of the strong law of large numbers for random variables. We also give two examples which show the sharpness of our results.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of real numbers with $0 < b_n \uparrow \infty$. Then $\{a_n X_n, n \geq 1\}$ is said to obey the general strong law of large numbers (SLLN) with norming constants $\{b_n, n \geq 1\}$ if

$$\frac{\sum_{i=1}^{n} a_i X_i}{b_n} \to 0 \quad \text{almost surely (a.s.).} \tag{1.1}$$

When $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, the SLLNs of the form (1.1) have been established by many authors. The special cases of (1.1) are the Kolmogorov SLLN ($a_n = 1, b_n = n$) and the Marcinkiewicz–Zygmund SLLN ($a_n = 1, b_n = n^{1/p}, 0 < p < 2$). Jamison et al. (1965) obtained sufficient conditions for (1.1) when $b_n = \sum_{i=1}^{n} a_i$. For more general weights, Jajte (2003) established sufficient conditions for (1.1). On the other hand, Sung (2011) gave sufficient conditions for (1.1) under the condition that $\{X_n, n \geq 1\}$ is a sequence of dependent random variables satisfying some moment inequalities.

In this paper, we will focus on the random variables which have no conditions on the joint distributions of the $\{X_n\}$. It is not assumed that $E|X_n| < \infty$ for all $n \geq 1$.

When $a_n = 1$ for all $n \geq 1$, Martikainen and Petrov (1980) proved a SLLN for random variables.

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Theorem 1.1 (*Martikainen and Petrov, 1980*). *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables. Let $\{b_n, n \geq 1\}$ be a sequence of positive numbers with $0 < b_n \uparrow \infty$. If*

$$b_n \sum_{i=n}^{\infty} \frac{1}{b_i} = O(n) \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} P(|X| > b_n) < \infty, \tag{1.3}$$

then $\sum_{i=1}^{n} X_i/b_n \to 0$ a.s.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a constant $0 < D < \infty$ such that

$$P(|X_n| > x) \leq DP(|X| > x) \quad \text{for all } x > 0 \text{ and } n \geq 1.$$

Adler and Rosalsky (1987) extended the SLLN of Martikainen and Petrov (1980) to weighted sums.

Theorem 1.2 (*Adler and Rosalsky, 1987*). *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of real numbers satisfying $0 < b_n \uparrow \infty$ and*

$$\max_{1 \leq i \leq n} \frac{b_i}{|a_i|} \sum_{i=n}^{\infty} \frac{|a_i|}{b_i} = O(n). \tag{1.4}$$

If

$$\sum_{n=1}^{\infty} P(|a_n X| > b_n) < \infty, \tag{1.5}$$

then the SLLN (1.1) holds.

If $a_n = 1$ for all $n \geq 1$, then conditions (1.4) and (1.5) are identical to conditions (1.2) and (1.3), respectively.

The Marcinkiewicz–Zygmund SLLN states that if $\{X, X_n, n \geq 1\}$ is a sequence of i.i.d. random variables such that $E|X|^p < \infty$ for some $0 < p < 2$ and $EX = 0$ if $1 \leq p < 2$, then (1.1) holds with $a_n = 1$ and $b_n = n^{1/p}$. It is well known that the independence hypothesis is not necessary when $0 < p < 1$. This fact can also be obtained from Theorem 1.1 noting that

$$\sum_{i=n}^{\infty} \frac{1}{i^{1/p}} = O(n^{1-1/p}) \quad \text{if } 0 < p < 1, \tag{1.6}$$

and $\sum_{n=1}^{\infty} P(|X| > n^{1/p}) < \infty$ is equivalent to $E|X|^p < \infty$.

Rosalsky and Stoica (2010) obtained a SLLN for identically distributed random variables under a more general condition than (1.6).

Theorem 1.3 (*Rosalsky and Stoica, 2010*). *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables with $E|X|^p < \infty$ for some $0 < p < 1$. Let $\{b_n, n \geq 1\}$ be a sequence of positive numbers satisfying $0 < b_n \uparrow \infty$ and*

$$\sum_{i=n}^{\infty} \frac{1}{b_i} = O\left(\frac{1}{b_n^{1-p}}\right).$$

If (1.3) holds, then $\sum_{i=1}^{n} X_i/b_n \to 0$ a.s.

Recently, Liao and Rosalsky (2013) proved a SLLN for weighted sums of random variables.

Theorem 1.4 (*Liao and Rosalsky, 2013*). *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $0 < p < 1$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of real numbers satisfying $0 < b_n \uparrow \infty$ and*

$$\frac{|a_n|}{b_n} = O\left(\frac{1}{n^{1/p}}\right).$$

Then the SLLN (1.1) holds.