Statistics and Probability Letters 118 (2016) 94–99

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Number of critical points of a Gaussian random field: Condition for a finite variance

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ARTICLE INFO

Article history:
Received 18 February 2016
Received in revised form 20 June 2016
Accepted 22 June 2016
Available online 2 July 2016

Keywords:
Gaussian field
Critical point
Geman condition
Crossing

ABSTRACT

We study the number of points where the gradient of a stationary Gaussian random field restricted to a compact set in $\mathbb{R}^d$ takes a fixed value. We extend to higher dimensions the Geman condition, a sufficient condition on the covariance function under which the variance of this random variable is finite.

0. Introduction

Let d be a positive integer and let $X : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a stationary Gaussian random field. We assume that almost every realization is of class $\mathcal{C}^2$ on $\mathbb{R}^d$. For T, a compact set in $\mathbb{R}^d$ and for any $v \in \mathbb{R}^d$, we consider

$$N^{X'}(T, v) = \#\{t \in T : X'(t) = v\},$$

where # denotes the cardinality of the set. For $v = 0$, it is nothing but the number of critical points of X in T. In this paper, we establish a sufficient condition on the covariance function r of the random field X in order that $N^{X'}(T, v)$ admits a finite variance.

The existence of the second moment of $N^{X'}(T, v)$ has been studied since the late 60s, first in dimension one and for a level v equal to the mean. Cramér and Leabetter (1965) were the first to propose a sufficient condition based on the covariance function r. If X satisfies some non-degeneracy assumptions, this simple condition requires that the fourth derivative $r^{(4)}$ satisfies

$$\exists \delta > 0, \quad \int_0^\delta \frac{r^{(4)}(0) - r^{(4)}(t)}{t}\, dt < +\infty.$$

It is known as the Geman condition for Geman proved some years after in Geman (1972) that it was not only sufficient but also necessary. The issue of the finiteness of the higher moments of $N^{X'}(T, 0)$ has also been discussed in many papers (see Belyaev, 1966; Cuzick, 1975; Malevich, 1985 for instance and references therein). Kratz and León generalized Geman's result in Kratz and León (2006) to the number of crossings of any level $v \in \mathbb{R}$ and also to the number of a curve crossings.

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http://dx.doi.org/10.1016/j.spl.2016.06.018
0167-7152/

Concerning the problem in higher dimensions, it has been an open question for a long time. Except an evocation of a similar question in Adler and Hasofer (1976) and an unacknowledged paper (Elizarov, 1985), which both give short and elliptical proofs, we could not find in the literature any sufficient condition on r for $N^{X'}(T,v)$ to be in $L^2(\Omega)$ without any condition on the isotropy of X. Note that in Malevich (1985), a partial answer is given through a condition on the spectral density and a positive answer is given in Estrade and León (in press) under the additional hypothesis that X is isotropic and of class $\mathcal{C}^3$.

The paper is organized as follows. Notations and assumptions are introduced in Section 1. Section 2 is devoted to preliminary estimations that use, as in dimension one, classical Rice formulas and Gaussian regression. The main result of this paper, namely Theorem 3.1, is proved in Section 3 with arguments that are specific to higher dimensions.

1. Notations and hypothesis

We denote by $r : t \mapsto \mathrm{Cov}(X(0), X(t))$ the covariance function of X. It is of class $\mathcal{C}^4$, since almost every realization of X is of class $\mathcal{C}^2$ on $\mathbb{R}^d$.

We fix an orthonormal basis of $\mathbb{R}^d$, according to the canonical scalar product that we denote by $\langle \cdot\,, \cdot \rangle$. We consider the partial derivatives of X and r computed on this basis. We write $(X'_i)_{1\le i\le d}$ and $(X''_{i,j})_{\substack{1\le i\le d\\1\le j\le d}}$ the partial derivatives of X of first and second orders, respectively, and r'_i, $r''_{i,j}$, $r^{(3)}_{i,j,m}$ and $r^{(4)}_{i,j,m,n}$ the partial derivatives of r, from order one to four, respectively. We refer to the gradient of X at t as $X'(t)$ and to the Hessian matrix of X at t as $X''(t)$. Similarly, we write $r''(t)$ the Hessian of r at t. We will sometimes denote by $r^{(3)}_{i,j}(t)$ the vector $(r^{(3)}_{i,j,m}(t))_{1\le m\le d}$ and by $r^{(4)}_{i,j}(t)$ the matrix $(r^{(4)}_{i,j,m,n}(t))_{\substack{1\le m\le d\\1\le n\le d}}$. We also use the same notation for $t \in \mathbb{R}^d$ and the column vector containing its coordinates.

In every space $\mathbb{R}^m$ (m is any positive integer), we denote by $\|\cdot\|$ the norm associated with the canonical scalar product. We use the standard notations $o(\cdot)$ and $O(\cdot)$ to describe the behavior of some functions in a neighborhood of zero.

We make the following assumptions on X, referred to as (**H**):

$$(\mathbf{H})\ \begin{cases} \text{almost every realization of } X \text{ is of class } \mathcal{C}^2, \\ \forall t\neq 0, \quad \mathrm{Cov}\left(X'(0), X'(t), (X''_{i,j}(0))_{1\le i\le j\le d}, (X''_{i,j}(t))_{1\le i\le j\le d}\right) \text{ is of full rank,} \\ r(0) = 1 \quad \text{and} \quad -r''(0) = I_d. \end{cases}$$

Note that the major assumptions in condition (**H**) are the first two ones. The third one is simply a consequence of the second one, which implies that $\mathrm{Var}[X(t)] = r(0) \neq 0$ and that the covariance matrix of $X'(0)$ is not degenerate. As a result, following the arguments given in the proof of Lemma 11.7.1 in Adler and Taylor (2007), we may assume that $r(0) = 1$ and $-r''(0) = I_d$. It makes the intermediate proofs and computations easier, but the main result of our paper remains true if we remove it. Note that we have reduced the problem to the case of a field with uncorrelated first-order derivatives, which is far from the isotropic case.

We introduce the $d\times d$ matrix $\Theta(t)$ defined by $\Theta(t)_{m,n} = \langle r^{(4)}_{m,n}(0)t, t\rangle$, which satisfies the Taylor formula $r''(t) = -I_d + \frac{1}{2}\Theta(t) + o(\|t\|^2)$. We note that (**H**) implies that $\Theta(t)$ is invertible for any $t \neq 0$. Hence, in what follows, we denote by $\Delta(t)$ the inverse matrix of $\Theta(t)$. Besides, we also remark that $t\mapsto\Theta(t)$ and $t\mapsto\Delta(t)$ are homogeneous functions of respective degrees 2 and -2.

We fix a compact set T in $\mathbb{R}^d$, such that the boundary of T has a finite $(d-1)$-dimensional Lebesgue measure.

2. Preliminary results

This section consists in a generalization to higher dimensions of the first steps taken in dimension one to prove that the Geman condition implies the finiteness of the variance of $N^{X'}(T,v)$ (see, for instance, Cramér and Leabetter, 1965, Geman, 1972 or Kratz and León, 2006). We present these steps without proof, for more details see Estrade and Fournier (2015).

The well-known Rice formula gives a closed formula for the expectation of $N^{X'}(T,v)$ and also states that it is finite in our context (see Rice, 1945 and Longuet-Higgins, 1957 as historical references and Adler and Taylor, 2007 Theorem 11.2.1 for a general statement). So the variance of $N^{X'}(T,v)$ is finite if and only if its second-order factorial moment is finite. We introduce the notation $p_{0,t}$ for the probability density function of the Gaussian vector $(X'(0), X'(t))$, as well as the function

$$F(v,t) = \mathbb{E}\left[|\det X''(0)\ \det X''(t)|\,\middle|\,X'(0) = X'(t) = v\right]; \quad v, t\in\mathbb{R}^d.$$

Under the stationarity assumption and hypothesis (**H**), the following Rice formula for the second factorial moment holds whether both sides are finite or not (see Azaïs and Wschebor, 2009 Theorem 6.3 or Adler and Taylor, 2007 Corollary 11.5.2):

$$\mathbb{E}[N^{X'}(T,v)(N^{X'}(T,v)-1)] = \int_{\mathbb{R}^d} |T\cap(T-t)|\,F(v,t)\,p_{0,t}(v,v)\,dt,$$

where $|T\cap(T-t)|$ is the Lebesgue measure of $T\cap(T-t)$. It is easy to prove that $p_{0,t}(v,v)$ is bounded by $\|t\|^{-d}$ (up to a constant factor) for t in a neighborhood of zero. Consequently, the above formula allows us to give a simple criterion for $N^{X'}(T,v)$ to be square integrable.