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Transport processes with random jump rate

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ABSTRACT

The aim of this paper is to study transport processes with random jump rate, i.e. mixed transport processes. We introduce and construct such processes by means of the approach based on dynamical systems. Furthermore, if our models evolve linearly, a strong large number law and a functional central limit theorem hold.

1. Introduction

Transport processes are stochastic dynamical systems. More precisely, evolutionary systems that change their mode of evolution following a stochastic mode are transport processes. The motion of a particle whose velocity performs jumps of random length at random times represents the prototype of transport process. The simplest example is the telegraph process: a particle starting to move on the real line with constant velocity $c > 0$, changes it at a random time, given by the arrival of a Poisson process, to $-c$; that is, the velocity of the particle at time t is $c(-1)^{N(t)}$ where $(N(t))_{t\geq 0}$, is a (homogeneous) Poisson process. Transport processes are stochastic models in random media. In the above example, the Poisson process represents the random medium.

Usually, the transport processes are introduced by taking into account a Markov process $(X(t), V(t))_{t\geq 0}$, where $X(t)$ is the state of the evolution system at time t, while the driving process $V(t)$ represents the random perturbation of outer media. For instance, for the telegraph model introduced above, $X(t)$ is the position at time $t > 0$ reached from the particle and $V(t)$ is the velocity-jump process. The transport processes have been studied by several authors. For instance, in Kurtz (1973) and Baggett and Stroock (1974) some results on convergence of perturbed operator semigroups allow to obtain limit theorems for such $(X(t))_{t\geq 0}$. In Stroock (1974) the process is constructed by means of the martingale problem, while a very general approach in the study of the diffusion approximation for these processes can be found in Papanicolaou (1975) and Bensoussan et al. (1979). Transport processes in bounded domains with different reflection principles have been studied in Costantini (1991) and Costantini and Kurtz (2006). Very close to the main object of this paper are the so-called piecewise-deterministic Markov processes; for a review of this topic the reader can consult Davis (1993) and Azaïs et al. (2014).

An abstract presentation of the transport processes as operator-valued stochastic processes in Banach spaces can be found in Pinsky (1991), Swishchuk (1997) and Koroliuk and Limnios (2005). This general class of stochastic processes is called "random evolutions". Random evolutions can be applied to describe several phenomena such as: displacement of

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microorganisms (see e.g. Stroock, 1974; Hillen and Othmer, 2000), wave propagations (see e.g. Bal et al., 2000), insurance risks (see e.g. Davis, 1993; Swishchuk, 2000), evolution of asset prices (see e.g. Swishchuk, 2000).

A central role in the definition of transport processes is played by the intensity of the jumps caused by the random medium. Usually, this quantity is introduced by a Poisson-type process $\mathcal{N}$; that is the waiting times for changes of the stochastic mode are exponentially distributed with a deterministic jump rate. Nevertheless, it seems reasonable to consider the jump rate as a random variable or equivalently as a realization of a random variable with known distribution. For instance, in the financial applications of the random evolutions the jump rate could depend by the subjective opinion or the belief of the financial operators. In this context the probability distribution of the intensity rate can be regarded as a prior distribution. Therefore, in this paper, we deal with *transport processes with random jump rate* or for shortness *mixed transport processes* (MTP). We choose $\mathcal{N}$ as a mixed Poisson process which, roughly speaking, is a mixture, with respect to a mixing random variable representing the random intensity, of homogeneous Poisson processes. Therefore, the jump rate is randomly chosen and remains fixed throughout the time evolution of the process. Furthermore, in this case the waiting times before jumping to a new mode are not exponentially distributed. For this reason the couple $(X(t), V(t))_{t\geq 0}$ is not jointly Markovian.

We recall some standard notations on the functional spaces used in this paper.

- Given a set E, let us indicate by $\mathcal{B}(E)$ the Borel algebra on E, i.e. the σ-algebra generated by open sets.
- Let $C_0(E)$ be the space of bounded continuous functions on E such that one has that $\lim_{|x|\to\infty} f(x) = 0$; namely for each $\varepsilon > 0$ there exists a compact set K_ε such that $|f(x)| < \varepsilon$ for each $x \notin K_\varepsilon$. The norm is given by $\|f\| = \sup_x |f(x)|$.
- Let $C_E[0,\infty)$ (resp. $C_E[0,1]$) be the space of continuous function on $[0,\infty)$ (resp. $[0,1]$) taking values on E. $C_E[0,\infty)$ (resp. $C_E[0,1]$) with the metric leading to the topology of uniform convergence on the compact sets (resp. topology of uniform convergence) is a separable complete metric space.
- Let $D_E[0,\infty)$ (resp. $D_E[0,1]$) be the space of right-continuous functions having left-limits on $[0,\infty)$ (resp. $[0,1]$) taking values in E. $D_E[0,\infty)$ (resp. $D_E[0,1]$) with the Skorokhod topology induced by the metric (16.4), p.168, (Billingsley, 1999) (resp. (12.16), p.125, Billingsley, 1999), is a complete separable metric space.

The paper is organized as follows. Section 2 represents a brief introduction to the mixed Poisson process. In Section 3, we provide the probabilistic construction of MTP and some properties of the processes. In Section 4, we consider MTP where the evolution of the system is governed by linear functions. In this case we are able to obtain some asymptotic results for the process like strong law of large numbers and a functional central limit theorem. Furthermore, we propose a random model describing the dynamics of the price of the risky assets. The proofs are collected in the last section.

2. Mixed Poisson process

In this section we introduce the mixed Poisson process and recall its basic properties. A detailed presentation of this process can be found in the monographs Grandell (1997) and Rolski et al. (1999).

Let $\mathcal{I}$ be the set of all (counting) functions $\nu = (\nu(t))_{t\geq 0}$ such that: $\nu(0) = 0$; $\nu(t)$ is an integer for all $t < \infty$; $\nu(\cdot)$ is non-decreasing and right-continuous. Let $\mathcal{B}(\mathcal{I}) = \sigma(\nu(t) \leq y; \nu \in \mathcal{I}, t \geq 0, y < \infty)$. The set belonging on $\mathcal{B}(\mathcal{I})$ is, for instance, of the following form:

$$B = \{\nu \in \mathcal{I}; \nu(t_1) = k_1, \nu(t_2) = k_2, \dots, \nu(t_n) = k_n\}, \tag{2.1}$$

with $n = 1, 2, \dots, 0 = t_0 < t_1 < t_2 < \cdots < t_n$ and $0 = k_0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$. Let Λ be a positive random variable with probability distribution $\mathbb{P}$ on $((0,\infty), \mathcal{B}(0,\infty))$. Let us denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. Now, we provide the following definition (see Grandell, 1997).

Definition 1. An integer-valued random measure $\mathcal{N} = (\mathcal{N}(t))_{t\geq 0}$ on $(S, \mathcal{S}, P)$, is called *mixed Poisson process*, with respect to Λ, if its distribution is a probability measure $P^{\mathcal{N}}$ on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ given by

$$P^{\mathcal{N}}(B) = P\mathcal{N}^{-1}(B) = \int_{(0,\infty)} Q_\lambda(B)\mathbb{P}(\mathrm{d}\lambda), \quad B \in \mathcal{B}(\mathcal{I}), \tag{2.2}$$

where

$$Q_\lambda(B) = \prod_{j=1}^{n} \frac{(\lambda(t_j - t_{j-1}))^{k_j - k_{j-1}}}{(k_j - k_{j-1})!} e^{-\lambda(t_j - t_{j-1})},$$

for all the sets B of type (2.1).

The probability measure $P^{\mathcal{N}}(B)$ is determined and well-defined for each $B \in \mathcal{B}(\mathcal{I})$. Indeed $(Q_\lambda(\cdot))_{\lambda\geq 0}$ represents the conditional probability measure on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ of $\mathcal{N}$ given Λ (see Grandell, 1997). Hence, the above definition leads to the following interpretation of a mixed Poisson process. $\mathcal{N}$ can be seen as a mixture of homogeneous Poisson processes; in other words, first a realization λ of a positive random variable Λ is generated and conditioned upon that realization, $\mathcal{N}$ is a Poisson process with intensity λ. Therefore Λ is the mixing random variable while $\mathbb{P}$ represents the mixing distribution. Some examples of mixed Poisson processes are reported in Grandell (1997) and Rolski et al. (1999).