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Persistently unbounded probability densities

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ABSTRACT

The paper provides examples of how to construct probability densities whose convolution powers are all unbounded. This persistent form of unboundedness is related to a premise in a well-known local central limit theorem.

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1. Introduction

In probability theory, roughly spoken, a central limit theorem (CLT) states that, given certain conditions on their moments and mutual dependence, the arithmetic mean of a sufficiently large number of iterates of random variables will be approximately normally distributed, irrespective of their underlying distribution. To set some notation in this, suppose that $X_1, X_2, X_3, \ldots$ is a sequence of random variables with existing mean and variance. Define for all $n = 1, 2, 3, \ldots$ the random variable Y_n as

$$Y_n = \frac{X_1 + X_2 + \cdots + X_n}{n}. \tag{1}$$

Let Z_n be the variable Y_n in a standardised form, that is:

$$Z_n = \frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\mathbb{V}\mathrm{ar}(Y_n)}}. \tag{2}$$

A most basic CLT states that, if the sequence $n = 1, 2, 3, \ldots$ is i.i.d., the sequence of cdf's F_{Z_n} converges pointwise to the cdf Φ of the standard Gaussian distribution. The limit function Φ being continuous, it can be proved that this convergence is actually uniform on $\mathbb{R}$ (see for example Pestman, 2009; Petrov, 1976; Rényi, 2007).

Now suppose that one is in the frequently assumed scenario where the X_k have a common density f. Then the Z_n also have a density, which we will denote by f_{Z_n}. This density can be expressed in terms of convolution powers of f, that is to say, powers of the form

$$f^{*n} = \underbrace{f * f * \cdots * f}_{n \text{ factors}}. \tag{3}$$

In terms of such powers f_{Z_n} may be expressed as:

$$f_{Z_n}(z) = \sigma\sqrt{n} f^{*n}\big(\sigma\sqrt{n}\, z + n\mu\big). \tag{4}$$

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In the above μ and σ^2 stand for the mean and variance of the density f. One may wonder under which conditions the sequence of densities f_{Z_n} converges in some sense to the density φ of the standard Gaussian distribution. Results of this type are called *local central limit theorems*. Such theorems can, roughly spoken, be classified as to the mode of convergence of the f_{Z_n}. For example, in Prokhorov (2016) a local CLT is established that assures, under suitable conditions, the f_{Z_n} to converge in mean to φ. In Rango Rao and Varadarajan (1960) a similar theorem assures pointwise convergence almost everywhere as to this. In this paper we will focus on a local CLT, proved by B.V. Gnedenko (see Gnedenko, 2016), that assures uniform convergence of the f_{Z_n} to φ. Among the premises in this theorem one finds a boundedness condition on the convolution powers f^{*n}. In the following, functions for which for every n the convolution power f^{*n} is an unbounded function will be called *persistently unbounded*. A function that fails to be persistently unbounded will be called *eventually bounded*. Hence, a function f is eventually bounded if there exists an integer n such that (3) is a bounded function. For the convolution product of two integrable functions f and g one generally has (see Rudin, 1987; Schwartz, 2008) the following inequality

$$\|f * g\|_\infty \leq \|f\|_\infty \|g\|_1. \tag{5}$$

It follows from this that, if f^{*n} is bounded for some n, then f^{*m} is bounded for all $m \geq n$. The concept of eventual boundedness plays a crucial role in Gnedenko's CLT:

Theorem 1. *Let $n = 1, 2, 3, \ldots$ be an i.i.d. sequence of random variables with existing expectation and variance. Suppose that the X_k have a common density f. Then the densities f_{Z_n} converge uniformly to the density of the standard Gaussian distribution if and only if f is eventually bounded.*

A proof of this result can be found for example in Gnedenko (2016) or Rényi (2007). Note that, under the premise of a finite mean and variance for the density f, the classical CLT guarantees that

$$\lim_{n\to\infty} \|F_{Z_n} - \Phi\|_\infty = 0. \tag{6}$$

Theorem 1 states that, in the case of an eventually bounded density f, one has

$$\lim_{n\to\infty} \|f_{Z_n} - \varphi\|_\infty = 0. \tag{7}$$

If f fails to be eventually bounded, then, because of (4), the density f_{Z_n} is unbounded for all n. The standard Gaussian density φ being bounded, one necessarily has

$$\|f_{Z_n} - \varphi\|_\infty = +\infty \quad \text{for all } n. \tag{8}$$

Hence there is a striking dichotomy as to convergence of the f_{Z_n}: one either has (7) or (8) and eventual boundedness of f is conclusive in this. Note that the scenario of (8) does not exclude weaker forms of convergence, such as converge in mean or pointwise convergence almost everywhere (see Prokhorov, 2016; Rango Rao and Varadarajan, 1960).

Let us focus now a bit on the concept of eventual boundedness. A bounded density is, of course, always eventually bounded. An eventually bounded density is, however, not always bounded:

Example. Define for every $a > 0$ the function f_a as

$$f_a(x) = \begin{cases} \dfrac{1}{\Gamma(a)} x^{a-1} e^{-x} & \text{if } x > 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{9}$$

These functions, being gamma-densities, satisfy the following permanence property as to convolution:

$$f_a * f_b = f_{a+b}. \tag{10}$$

See for example Pestman (2009) or Schwartz (2008) for a proof of the above. In particular one has:

$$f^{*n} = f_{na}. \tag{11}$$

The functions f_a are unbounded for $0 < a < 1$ and bounded for $a \geq 1$. From the above it follows that an unbounded density f_a can be turned into a bounded density by raising it to a convolution power n with $na \geq 1$. Hence, unbounded gamma densities are eventually bounded. □

In mathematical analysis it is well-known that the convolution product $f * g$ of two integrable functions f and g usually shows more regularity than each of the components (see for example Rudin, 1976, 1987; Schwartz, 2008). This also applies when interpreting boundedness as a form of regularity. For that reason most densities encountered in daily statistical life are eventually bounded. Actually one may wonder how to construct examples of densities that fail to be eventually bounded, that is, persistently unbounded densities. The following section will be devoted to this.

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